Studying PDE's we will use 2 notations
for partial derivatives:

e.g. $u \equiv u(x,y,z)$

$$\frac{\partial u}{\partial x} \overset{def.}{=} \lim_{h \to 0} \frac{u(x+h,y,z) - u(x,y,z)}{h}$$

We also say $\underbrace{u_x}_{\text{u SUB X}} = \underbrace{\frac{\partial u}{\partial x}}_{\text{DU OVER DX}}$

(subscript)

Similarly $\frac{\partial u}{\partial y} = u_y = \lim_{h \to 0} \frac{u(x,y+h,z) - u(x,y,z)}{h}$

... $\frac{\partial u}{\partial z} = u_z = \dots$

Partial derivatives of higher order

e.g. $u_{xz} = \frac{\partial^2}{\partial x \partial z} u = \frac{\partial}{\partial x} \frac{\partial u}{\partial z}$

"A priori" looks like there are many 2^{nd} order derivatives:

$u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xz}, u_{yz}, u_{yx}, u_{yz}, u_{zy}, u_{zx} \dots$

<u>Theorem (Schwarz)</u>

If $u \equiv u(x_1, x_2, \dots, x_n)$ function of n variables and
$u \in C^2(\Omega)$ $\Omega \subseteq \mathbb{R}^n$ then $\boxed{u_{x_i x_j} = u_{x_j x_i}}$

Actually this is true for any order m of partial differentiatio[n]
provided $u \in C^m(\Omega)$ [In this course, $m = 1, 2$]
Namely: mixed partial derivatives <u>do not</u> depend on the
order of differentiation if $u \in C^m(\Omega)$

examples: $u(x,y) = x^2 y + \sin(x+y)$

$u_x = 2xy + \cos(x+y)$ $u_y = x^2 + \cos(x+y)$

$u_{xx} = 2y - \sin(x+y)$

$u_{yy} = \dots$

SCHWARZ $\begin{cases} u_{xy} = 2x - \sin(x+y) \\ u_{yx} = 2x - \sin(x+y) \end{cases}$

If the 2^{nd} order partial derivatives of $u(x,y)$ are continuous there are only 3 of them u_{xx}, u_{yy}, u_{xy} $(u_{xy} = u_{yx})$

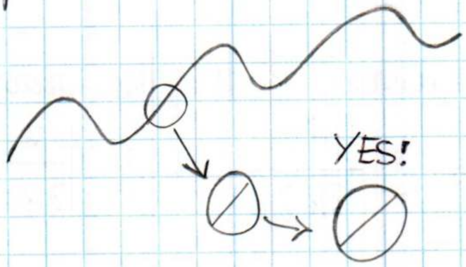
N.B. <u>Not true</u> in general if u_{xy} and u_{yx} are not continuous.

Another property that requires the continuity of partial derivatives is <u>differentiability</u>.

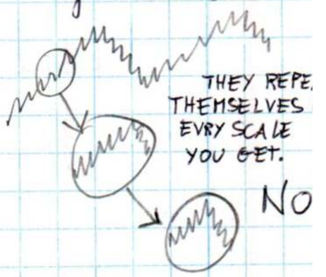
<u>N.B.</u> In one variable a function is differentiable at $x = x_0 \iff f'(x_0)$ exists (finite).

A function is differentiable at a point if there is a "good" linearization of it at that point.

Roughly speaking: a function is differentiable if you can linearize its enlargements.



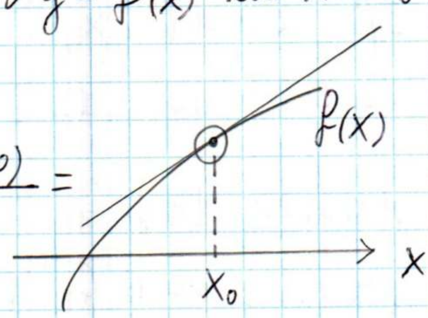
fractals:



Assume $\lim_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$ exists finite

we call this number $f'(x_0)$ and we know that this defines the slope of the tangent line to the graph $y = f(x)$ at $x = x_0$

The line is $y - f(x_0) = f'(x_0)(x - x_0)$



To say (assumption) $\lim_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$ is <u>equivalent</u> to

$$\boxed{\frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + r(x) \quad \text{with} \quad \lim_{x \to x_0} r(x) = 0}$$

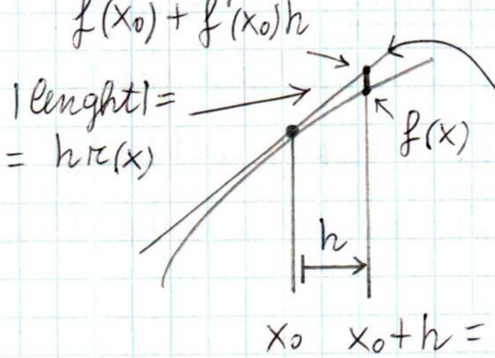
(SCRITTURA FUORI DEL SEGNO DI LIMITE)

("writing (notation) out of the limit sign")

$$f(x_0+h) - f(x_0) = f'(x_0)h + h\,r(x)$$

$$\underbrace{f(x_0+h)}_{x} = \underbrace{f(x_0) + f'(x_0)\,h}_{} + h\,r(x) \qquad (x-x_0) = h$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + (x-x_0)\,r(x)$$



$$y = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{}$$

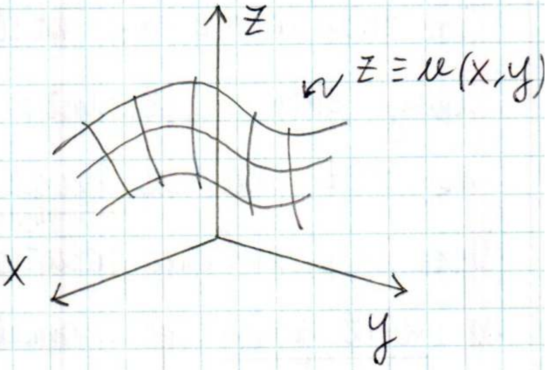
tangent line at $x = x_0$

Taylor polynomial of order 1

(linearization of $f(x)$ at $x = x_0$)

Differentiability (which is automatic in 1 variable, not in $n \geq 2$ variables) means that the discrepancy $h\,r(x)$ between $f(x)$ and the value on the tangent line (on x_0) computed at x, not only goes to 0 as $h = x - x_0 \to 0$, but it goes to 0 faster than h.

In two variables, let $u \equiv u(x,y)$; the tangent plane to the surface is:



$$\boxed{z - u(x_0,y_0) = u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)}$$

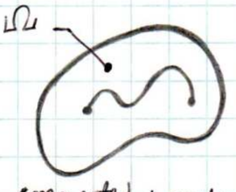
it might happen that $u_x(x_0,y_0)$ and $u_y(x_0,y_0)$ exist finite but the plane is not a good linearization.

Def. $u \equiv u(x,y)$ is differentiable at $P \equiv (x_0,y_0)$ if

$$u(x,y) - \{u(x_0,y_0) + u_x(x_0,y_0)(x-x_0) + u_y(x_0,y_0)(y-y_0)\} \to 0$$

as $(x,y) \to (x_0,y_0)$ faster than $\sqrt{(x-x_0)^2 + (y-y_0)^2}$ (distance between (x,y) and (x_0,y_0)). Generalizes to n variables.

Theorem if $u \equiv u(x_1, x_2, \dots, x_n) \in C^1(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ then the function u is differentiable at all points of Ω.



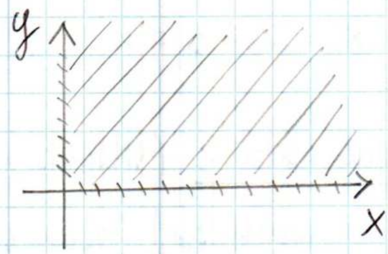
connected: any two points in Ω can be joined by a polygonal line in Ω.

From now on we will use the capital greek letter Ω to denote a region in $\mathbb{R}^n$ (open, connected subset of $\mathbb{R}^n$)

Open means that for each $P \equiv (x_1, x_2, \dots x_n)$ there exists a ball of center P and radius $r > 0$ entirely contained in Ω. (Roughly speaking: a region without its border).

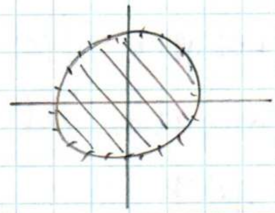
Examples of regions $\Omega \subseteq \mathbb{R}^2$

$\Omega \equiv \{(x,y) : x > 0, y > 0\}$
(unbounded region)



$\Omega \equiv \{(x,y) : x^2 + y^2 < 25\}$

Disc of center $(0;0)$, radius $r = 5$



In this course we will solve PDE's in regions Ω. Without any extra costraints (which are usually "boundary conditions") a PDE has a huge space of solutions. With suitable constraints there is a unique solution. We say that a PDE problem with B.C. is well posed if a small change at the boundary produces a small change in the solution.

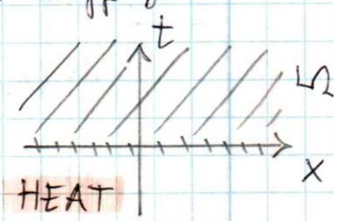
An example of what will follow...

$u \equiv u(x,t)$ unknown function

$u_t = u_{xx}$ PDE (linear, second order, constant coeff's

$(\partial u / \partial t = \partial^2 u / \partial x^2)$ parabolic)

Let's solve this in $\Omega \equiv \{(x,y) \in \mathbb{R}^2$ with $y > 0\}$



HEAT EQUATION

B.C. $u(x,0) = u_0(x) \longrightarrow$ GIVEN FUNCTION

sub-example $u_t = u_{xx}$ $u(x,0) = \dfrac{1}{1+x^2}$ solve in Ω

SECOND ORDER

Principle (vague but useful...) if we solve a PDE in a region Ω 1) if the boundary is a line: use Fourier Transform
2) if the boundary is a half line: use Laplace Transform
3) if the boundary is a segment or a circle (or a sector): use Fourier Series (or generalized orthogonal systems, e.g., orthogonal polynomials)

$u_0(x)$ could be the initial temperature of an infinite rod

$$\boxed{V(\xi,t) \overset{\text{def.}}{=} \int_{-\infty}^{+\infty} u(x,t)\, e^{-ix\xi}\, dx}\ .$$

PARTIAL FOURIER TRANSFORM (F.T. of our unknown function $u(x,t)$ with respect to the variable x, t fixed)

$$V_t(\xi,t) = \int_{-\infty}^{+\infty} u_t(x,t)\, e^{-i\xi x}\, dx \overset{\text{because of our PDE}}{=} \int_{-\infty}^{+\infty} u_{xx}(x,t)\, e^{-i\xi x}\, dx =$$

$$\overset{\text{derivative property of F.T.}}{=} (i\xi)^2 \int_{-\infty}^{+\infty} u(x,t)\, e^{-i\xi x} dx = -\xi^2\, V(\xi,t)$$

we get $\dfrac{\partial V}{\partial t} = -\xi^2 V$ (separable variable ODE)

$$\int \frac{dV}{V} = -\xi^2 \int dt \qquad \log|V| = -\xi^2 t + c$$

sol. ODE for V in variable t

$$|V| = \underbrace{e^{-\xi^2 t}}_{\text{always positive}}\ \underbrace{e^{c}}_{\text{positive arb. constant}}$$

so we can remove the abs. val. $\Rightarrow V = \underbrace{C'}_{\text{arbitrary real constant}} e^{-\xi^2 t}$

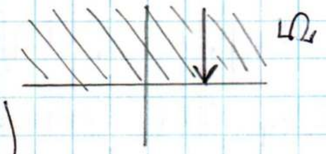
$C' = V(\xi,0)$ (value at $t=0$ of our F.T.)

$$V(\xi,t) = V(\xi,0)\, e^{-\xi^2 t}$$

<u>Define</u> $V_0(\xi) = [u_0(x)]^{\wedge}(\xi)$ where $u_0(x)$ is a <u>given</u> function at the boundary.

in the sub-example we would have $V_0(\xi) = \left(\frac{1}{1+x^2}\right)^\wedge(\xi)$

Now observe that $V(\xi, t) \to V_0(\xi)$ for $t \to 0^+$



$\Rightarrow V(\xi, t) = V_0(\xi)\, e^{-\xi^2 t}$ (F.T. in x variable of our solution $u(x,t)$)

Inversion formula: FOR THIS FUNCTION USE THE I.F.T.: $F(F(f))(x) = 2\pi f(-x)$

$$\Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\, V_0(\xi)\, e^{-\xi^2 t}\, d\xi = \frac{1}{2\pi}\, u_0 * \left(e^{-\xi^2 t}\right)^\wedge(x)$$

The inverse F.T. of product is a convolution

N.B. $\left(e^{-t\xi^2}\right)^\wedge(x) = \sqrt{\frac{\pi}{t}}\, e^{-x^2/4t}$ (see page 30)

$$u(x,t) = u_0 * \sqrt{\frac{\pi}{4\pi^2 t}}\, e^{-x^2/4t} = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} u_0(y)\, e^{-\frac{(x-y)^2}{4t}}\, dy$$

"explicit" formula for an unique solution.

Whatever it is the boundary function $u_0(x)$ given, the solution to our problem is a convolution of this times a Gaussian function $\Rightarrow$ the solution is unique, small changes in $u_0(x)$ produce small changes in the solution.

Next time we'll see how to classify 2nd order linear PDE's into elliptic, parabolic, hyperbolic.

We will see that the operator Δu comes up often (Laplace Operator, Laplacian) Def. if $u \equiv u(x_1, x_2, \dots, x_n)$

$$\Delta u = \frac{\partial^2}{\partial x_1^2} u + \frac{\partial^2}{\partial x_2^2} u + \dots + \frac{\partial^2}{\partial x_n^2} u = u_{x_1x_1} + u_{x_2x_2} + \dots + u_{x_nx_n}$$

Def. If u satisfies the PDE $\boxed{\Delta u = 0}$ for $(x_1, x_2, \dots, x_n) \in \Omega \subseteq \mathbb{R}^n$ we say that u is harmonic in Ω.

In the case of functions of two variables $u \equiv u(x,y)$ there is a strict relationship between harmonic and holomorphic functions.

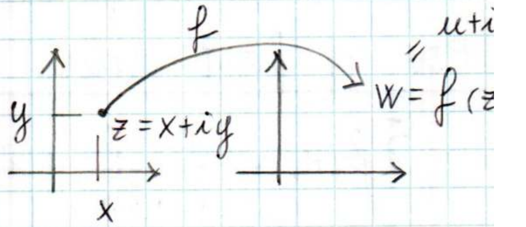
Def. Harmonic in 2 variables $\Delta u = 0$ in Ω

i.e. $u_{xx} + u_{yy} = 0$ for $(x,y) \in \Omega \subseteq \mathbb{R}^2$

Def. Holomorphic $f(z)=f(x+iy)$ $f(z)$ is holomorphic for $z=x+iy\in\Omega\subseteq\mathbb{C}$ if $\lim\limits_{h\to 0}\frac{f(z+h)-f(z)}{h}=f'(z)$ exists finite for $z\in\Omega$.

N.B. We have seen that holomorphic functions can be locally represented with power series (Taylor series) and there is analytic continuation. They have many other properties...

Observation $f(z):\Omega\to\mathbb{C}$ $(\Omega\subseteq\mathbb{C})$ $\quad f:\mathbb{R}^2\to\mathbb{R}^2$



Let's consider $z_0=0=f(z_0)$

$$f(0)=0$$

> $f(z)$ is holomorphic $\Longleftrightarrow$ it is differentiable as a function of two variables x,y. Namely $\exists\,\alpha,\beta\in\mathbb{C}$ such that
>
> (*) $f(z)=\alpha x+\beta y+r(z)\cdot z$ as $z\to 0$
>
> Actually $\alpha=\frac{\partial f}{\partial x}(0)$ $\quad\beta=\frac{\partial f}{\partial y}(0)$

Rem: $z+\bar z=(x+iy)+(x-iy)=2x=2\,\mathrm{Re}(z)$

$$z-\bar z=(x+iy)-(x-iy)=2iy=2\,\mathrm{Im}(z)$$

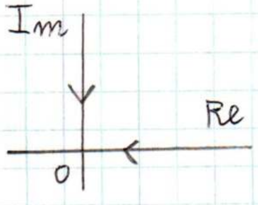
(*) equivalent to $\quad f(z)=\frac{\alpha-i\beta}{2}z+\frac{\alpha+i\beta}{2}\bar z+r(z)z$

$$\boxed{\text{Def. } \partial=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\qquad\bar\partial=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)}$$

$$\boxed{\frac{f(z)}{z}=\partial f(0)+\bar\partial f(0)\frac{\bar z}{z}+r(z)}$$

Note that if $z\in\mathbb{R}$ (then $\bar z=z$) $\Rightarrow \bar z/z=1$

if $z\in i\mathbb{R}$ (pure imaginary) (then $\bar z=-z$) $\Rightarrow \bar z/z=-1$



If $z\to 0$ first along the real axis, then along the imaginary axis; the only way for the two limits of $f(z)/z$ to coincide is $\bar\partial f(0)=0$

<u>**Theorem**</u> Suppose $f(z): \Omega \to \mathbb{C}$ is differentiable for $z \in \Omega$, then $f(z)$ is <u>holomorphic</u> $\Leftrightarrow$ $\boxed{\bar\partial f(z)=0}$, $\forall z \in \Omega$

$\bar\partial f(z)=0$ is a PDE (with complex constant coeff's) of 1^{st} order. It is called <u>Cauchy-Riemann equation.</u>

The holomorphic functions <u>are</u> the space of solutions of a specific PDE of first order.

$$\bar\partial f(z) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(x+iy)$$

<u>Another way</u> to look at this is to write $f(z) = f(x+iy) =$
$= u(x,y) + i\, v(x,y)$

$$\bar\partial f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(u(x,y) + i\, v(x,y)\right) \overset{\text{Cauchy Riemann}}{=} 0$$

$$\frac{1}{2}(u_x + i v_x) + \frac{i}{2}(u_y + i v_y) = 0$$

$$\frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y) = 0$$

$$\boxed{\text{Get} \begin{cases} u_x = v_y & (x,y) \in \Omega \\ u_y = -v_x & z = x+iy \in \Omega \end{cases}}$$

Cauchy Riemann System

<u>Verification in a couple of examples</u>

$f(z) = z^2$ is holomorphic in $\mathbb{C}$ (all $p(z)$ polyn. are holo. in $\mathbb{C}$)

$$z^2 = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$$

$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$

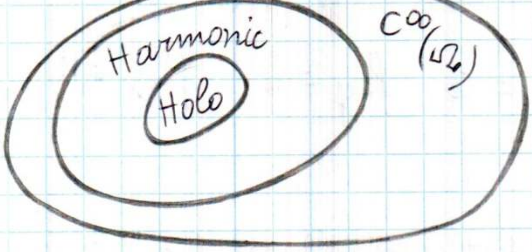
Other example $f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

$$= \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$$

$$u_x = e^x \cos y = v_y \qquad u_y = - e^x \sin y = - v_x$$

Observation $\boxed{\Delta f = f_{xx} + f_{yy} = 4 \partial \bar{\partial} f}$; from this it's easy to se

Theorem Holomorphic functions in $\Omega \subseteq \mathbb{C}$ are harmonic (in two variables x, y). Viceversa is false.



Remember that we have also seen that holomorphic functions (i.e. $C^1(\Omega)$) are also $C^\infty(\Omega)$.

Theorem if $f(z)$ is holomorphic in $\Omega \subseteq \mathbb{C}$, then $u(x,y) = \mathrm{Re}(f(z))$ and $v(x,y) = \mathrm{Im}(f(z))$ are (real) harmonic functions in x, y.

Remark If we choose two unrelated harmonic functions $a(x,y)$ and $b(x,y)$ usually it is not true that $a(x,y) + i\, b(x,y)$ is holomorphic. But it is (complex) harmonic.

Theorem Given $u(x,y)$ real-harmonic, there is a unique (except for an additive arbitrary constant) $v(x,y)$ real-harmonic such that $u(x,y) + i\, v(x,y)$ this is holomorphic.

An important property of harmonic functions (in particular holomorphic) is the Maximum Principle.

Def. $\bar{\Omega}$ closure of an open region is $\Omega \cup \{$accumulation points of $\Omega\}$

examples $\Omega \equiv \{(x,y) : x > 0, y > 0\}$ $\bar{\Omega} = \{(x,y) : x \geq 0 ; y \geq 0\}$

$\Omega \equiv \{(x,y) : x^2 + y^2 < 1\}$ $\bar{\Omega} \equiv \{(x,y) : x^2 + y^2 \leq 1\}$

If Ω is defined by inequalities, then in $\bar{\Omega}$ the strict inequalities become non-strict ($\geq \ldots \leq \ldots$)

<u>Maximum Principle</u> if $f(x,y)$ is <u>harmonic</u> in $\Omega \subseteq \mathbb{R}^2$

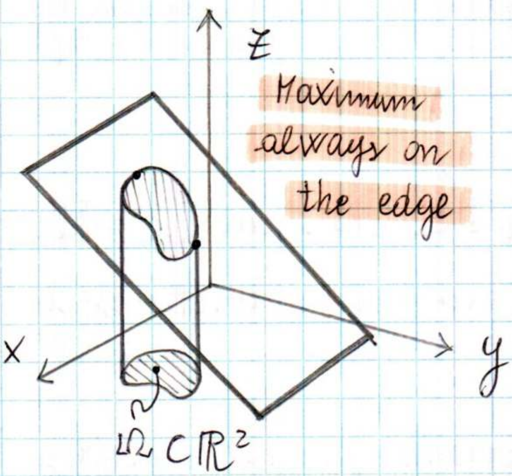
(i.e. $\Delta f = 0$ for $(x,y) \in \Omega$), then $\max_{(x,y)\in\bar{\Omega}} |f(x,y)|$ exists but cannot be attained in Ω. It is attained in the boundary
$f_{xx} + f_{yy} = 0$

[PROOF PAGE 72]

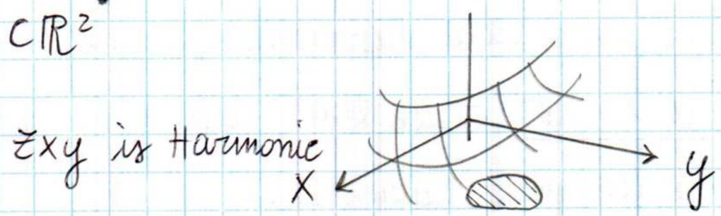
<u>Help for intuition:</u>
a very special case of harmonic functions are <u>planes</u> $f(x,y) = ax + by + c$

$f_x = a \quad f_y = b \quad f_{xx} = 0 \quad f_{yy} = 0$

$0 + 0 = 0$



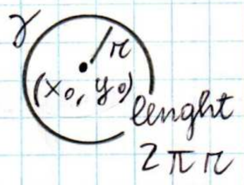
Actually (both for planes and general harmonic functions) there is also a minimum principle ...

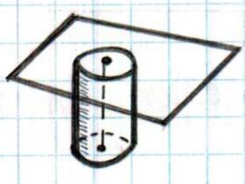


<u>Mean-Value Property</u>

If $f(x,y)$ is <u>harmonic</u> in $\Omega \subseteq \mathbb{R}^2$, then its value at $(x_0, y_0) \in \Omega$ is equal to the <u>integral average</u> of $f(x,y)$ taken on a circle centered at (x_0, y_0), namely $\frac{1}{2\pi r} \int_\gamma f(s)\, ds$

Where γ is the circle





If $n > 2$ (number of variables) there is no longer a strict connection between holomorphic and harmonic functions. But properties like Max. Princ. and Mean-Value remain true.