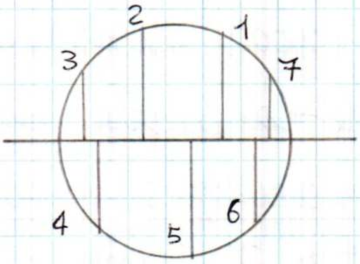
Generalized Leibnitz Test:

$\sum_{x=0}^{\infty} a_x b_x$ (SERIES); suppose that $\left|\sum_{x=0}^{N} a_x\right| \leq M$ independent of N and b_x satisfies (1)(2)(3) like regular Leibnitz test, <u>then</u> (*) converges.

<u>N.B.</u> In the regular L.T. we have $a_x = (-1)^x$ and clearly $\left|\sum_{x=0}^{N} (-1)^x\right| \leq 1 \quad \forall N$

<u>Exercise</u>: using this test show that $\sum_{x=1}^{\infty} \frac{\sin x}{x}$ converges

[P.19bis]



<u>Hint</u> choose $a_x = \sin x$, use Euler's Formula and the geometric series.

Euler's Formula: $e^{x+iy} = e^{x}(\cos y + i \sin y)$
$(x \in \mathbb{R}, y \in \mathbb{R})$

Theorem

Let $f \in C^{\infty}(I)$, where I is an interval $\subseteq \mathbb{R}$ of positive measure (possibly all of $\mathbb{R}$) (C^{∞} means that f has ∞ many derivatives $f', f'', f''', \ldots$ and they're all continuous)

Choose $x_0 \in I$ and write the Taylor's series of f:

$$\sum_{x=0}^{\infty} \frac{f^{(x)}(x_0)}{x!}(x - x_0)^x \quad (*)$$

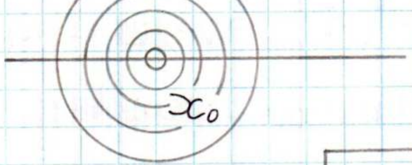
if the radius of convergence $R > 0$, then (*) converges to f in the interval $(x_0 - R; x_0 + R)$

<u>Def.</u> Function of this kind (C^∞ and $R>0$) are called **analytic functions**. If $\boxed{R=\infty}$ they are called **entire functions** (e.g. $\sin x$, $\cos x$, e^x are entire
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, $\sin x = x - \frac{x^3}{3} + \dots$,
$\cos x = 1 - \frac{x^2}{2!} + \dots$)

<u>Remark</u>: The set of analytic functions on $I \subseteq \mathbb{R}$ is much smaller than $C^\infty(I)$. There are C^∞ functions whose radius of convergence of the Taylor series is 0 everywhere !...

But (good news), if $R>0$, then not only f is real-analytic <u>but</u> it can be extended to a <u>complex analytic function</u> in an open disc in $\mathbb{C}$ of center x_0 and radius R (**analytic continuation**).

- If f is entire ($R=\infty$), analytic continuation can be performed completely via power series.
- If $R<\infty$, then analytic continuation is more complicated.



<u>Example</u>: $f(x) = \frac{1}{1+x^2}$ $\boxed{f \in C^\infty(\mathbb{R})}$ $f(x)$ satisfies the 1st cond. for analytic continuatio

(for $x \in \mathbb{R}$ the denominator $1+x^2>0$ in $f, f', f'', \dots$

choose $x_0 = 0$ and represent

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\cancel{x_0}\,0)}{k!}(x - \cancel{x_0}\,0)^k$$ {with $x_0 = 0$ we call it Mac Laurin series

To compute this P.S. there is a faster way than computing $f^{(k)}(x)$ directly

Use the G.S. $\frac{1}{1-t} = 1+t+t^2+\dots \quad |t|<1$ $(*)$

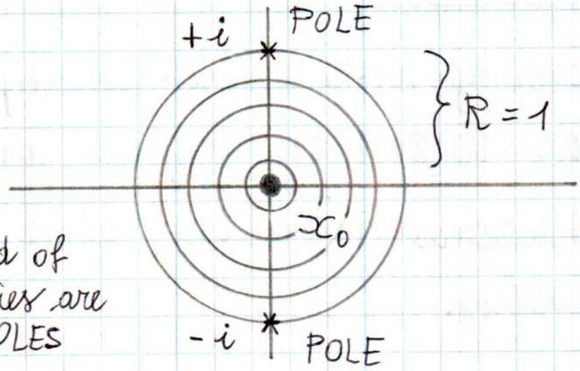
Set $t=-x^2$ in $(*)$ get $\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\dots$

$\boxed{R=1}$ (e.g. use ratio test $\left|\frac{a_{n+1}}{a_n}\right| = |x|^2 < 1$ for $|x|<1$)

so also the 2nd cond. is satisfied

The analytic continuation of $f(x) = \frac{1}{1+x^2}$ is $f(z) = \frac{1}{1+z^2}$ with $z \in \mathbb{C}$ and $f(z) = 1-z^2+z^4-z^6+\dots$

Note that den $= 1+z^2 = 0$ for $z=+i$ and $z=-i$



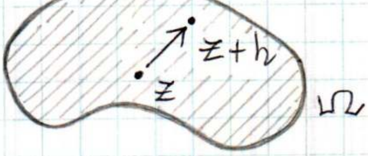
This kind of singularities are called POLES

The radius of convergence ($R=1$ in this example) coincide with the distance from x_0 (center of our P.S.) to the closest singularity of $f(z)$!

Note that $f(z) = \frac{1}{1+z^2}$ is defined in $\mathbb{C} - \{i; -i\}$ beyond the range of its P.S. centered at $z_i = 0$.

Moral: the process of analytic continuation is possible but more complicated to complete if $R < \infty$ (easy if $R=\infty$)

Def. A function $f(z)$ defined for $z \in \Omega \subseteq \mathbb{C}$ (open subset) is called holomorphic if its complex derivative $f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists finite for $\forall z \in \Omega$. (N.B. in this limit $h \in \mathbb{C}$, $z \in \Omega$, $z+h \in \Omega$)



<u>Theorem</u> (No proof) If $f(z)$ is holomorphic in Ω, then all of its derivatives exist $f'(z), f''(z), f'''(z), \dots$ in Ω <u>AND</u> is also complex analytic (with $R>0$), as long as we stay inside Ω.

<u>Remark</u> For REAL FUNCTIONS, being C^1 <u>does not</u> imply being C^2 etc. ... <u>AND</u> being C^∞ <u>does not</u> imply being real-analytic ($R>0$).

Compare (and be surprised!) with what happens for complex functions: if one derivative of $f(z)$ exists in an open set $\Omega \subseteq \mathbb{C}$, then f is C^∞ and analytic in Ω.

In this course we won't go deeply into complex analysis. There would be nice methods to exactly compute integrals (contour integration, Cauchy Theorems, Residues) and to exactly compute the sum of series.

<u>ex.</u> $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = \int_{-1}^{1} (1 - x^2 + x^4 - x^6 + \dots)\,dx$$

$$\left[\text{Arctg}x\right]_{-1}^{1} = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right]_{-1}^{1}$$

$$\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) - \left(-1 + \frac{1}{3} - \frac{1}{5} + \dots\right)$$

$$\cancel{2}\frac{\pi}{4} = \cancel{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \dots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

This computation is correct, but we should justify the $\int$ term by terms of the series...

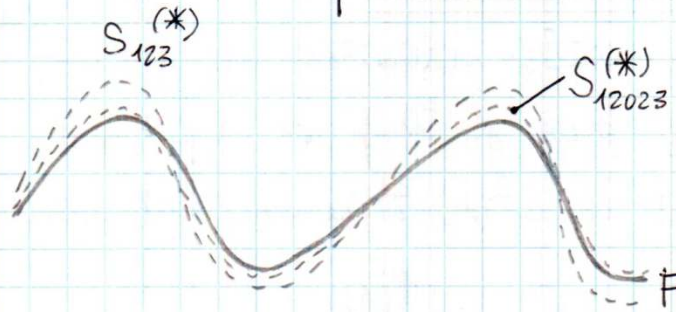
For series of functions $\sum_{n=0} f_n(x)$ (*) we have defined <u>pointwise</u> and <u>absolute</u> convergence.

There's another kind of convergence important in some theorems: UNIFORM CONVERGENCE.

<u>Def.</u> Let $S_N(x) = \sum_{n=0}^{N} f_n(x)$ be the partial sums of our series of functions; if

$$\lim_{N\to\infty} \sup \left| S_N(x) - F(x) \right| = 0$$

for all $x \in A$ we say that the series <u>converges uniformley</u> to $F(x)$ in A.



$F(x)$ is the "limit" function

$\sup | S_N(x) - F(x) |$ is the maximum discrepancy between partial sums & limit function

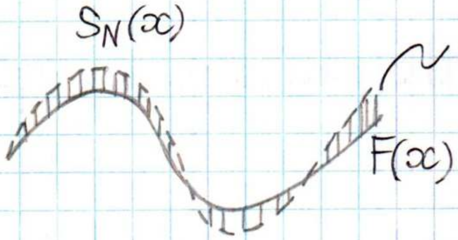
<u>Roughly Speaking</u>: if a series of functions $\sum_{n=0}^{\infty} f_n(x)$ converges absolutely and uniformley for $x \in A$, then we can operate on it "as if" it were a finite sum (commutative property, multiplication of two series, integration term by term, derivatives term by term...). More details will be added...

<u>By the way</u>, Power Series are a special case of series of functions with $f_n(x) = c_n (x - x_0)^n$

$L^1(A)$ and $L^2(A)$ spaces: we say that $f \in L^1(A)$ where $A \subseteq \mathbb{R}$ if $\int_A |f(x)|\,dx$ exists finite. We say that $f \in L^2(A)$ if $\int_A |f(x)|^2\,dx$ exists finite.

<u>N.B.</u> We could also consider for $1 \leq p < \infty$ $f \in L^p(A)$ if $\int_A |f(x)|^p\,dx$ exists finite.

$L^\infty(A)$ are (essentially) bounded functions on A.
Sometimes we will have series $\sum_{n=0}^{\infty} f_{(n)}(x) \to F(x)$ in $L^1(A)$ or in $L^2(A)$



in $L^1(A)$ means to consider the <u>area discrepancy</u> for the series to converge

<u>Def.</u> The $L^p(A)$ <u>norm</u> <u>of a function</u> is $\|f\|_p = \left(\int_A |f(x)|^p dx\right)^{1/p}$, $1 \leq p < \infty$ (we are mostly interested in $p = 1, 2$)

<u>Def.</u> The $L^p(A)$ <u>distance</u> from f to g is $\|f - g\|_p$ (we'll see later that in $L^2(A)$ there is also notion of angles).

<u>Def.</u> We say that $\sum_{n=0}^{\infty} f_n(x) \xrightarrow{L^p(A)} F(x)$ if its partial sums $\sum_{n=0}^{N} f_n(x) = S_N(x)$ satisfy $\lim_{N\to\infty} \|S_N(x) - F\|_p = 0$

<u>Remark</u> $\|f\|_\infty$ = "<u>essential supremum</u>" of f

If $\sum_{n=0}^{\infty} f_n(x) \xrightarrow{L^\infty} F$, this coincides (in most cases) with <u>uniform convergence</u>.

<u>O.D.E.</u> are diff. eqq. with <u>1 variable</u>. Generally have this form:

$$\boxed{F(x, y, y', \dots, y^{(n)}) = 0}$$

we are looking for functions $y = y(x)$ that (together with the derivatives $y', y'', \dots, y^{(n)}$) satisfy the eq.

<u>example:</u>
$2e^{y''} - y'\cos(y - x) - 1 = 0$
with solution $y(x) = x$
$y' = 1 \quad y'' = 0$

$2e^0 - 1 \cdot \cos(x - x) - 1 = 2 - 1 - 1 = 0$ ✓

{actually, there are many more solutions besides $y(x) = x$. We expect the general solution to contain 2 arbitrary constants because the order of this ODE is 2}