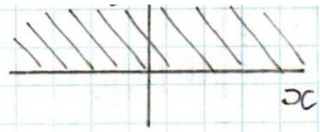
Review.

Ω 

Last class: $u_t = u_{xx}$ $\quad\Omega \equiv \{(x,t) : x \in \mathbb{R}, t > 0\}$

Harmonic functions $\Delta u = 0$

In 2 variables connections Holomorphic / Harmonic

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Th. (Cauchy-Riemann) $\left\{\begin{array}{l}\text{PDE of}\\ 1^{st}\text{ order}\end{array}\right.$

$f(z) = f(x+iy)$ is holomorphic in $\Omega \subseteq \mathbb{C} \iff \bar{\partial} f(z) = 0$ for $\forall z \in \Omega$

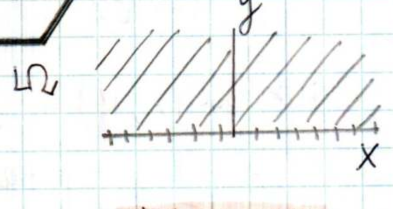
equivalent to: $f(z) = u(x,y) + i\,v(x,y)$

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

system of 2 PDE of first order in the 2 unknown functions $u(x,y)$ and $v(x,y)$

Theorem If $f(z)$ is holomorphic in Ω and $u(x,y) = \mathrm{Re}(f(z))$, $v(x,y) = \mathrm{Im}(f(z))$, then $u(x,y)$ and $v(x,y)$ are harmonic. Viceversa is false, i.e. if we take any pair of real harmonic functions $a(x,y)$ and $b(x,y)$ the complex function $a(x,y) + i\,b(x,y)$ is harmonic but not holomorphic.

In fact, given $u(x,y) = \mathrm{Re}(f(z))$ with f holomorphic there is a procedure to compute $v(x,y) = \mathrm{Im}(f(z))$ called the "harmonic conjugate" of $u(x,y)$.



ex. $\Delta u = 0 \qquad (u \equiv u(x,y))$

for $(x,y) \in \Omega \equiv \left\{(x,y) : \begin{array}{l} x \in \mathbb{R} \\ y > 0 \end{array}\right\}$

LAPLACE EQUATION

B.C. $u(x,0) = f(x)$ given function

(LAPLACE PROBLEM IN THE UPPER HALF PLANE)

$$\hat{u} \overset{def}{=} \int_{-\infty}^{+\infty} e^{-ix\xi} u(x,y)\, dx \quad \begin{pmatrix} \text{F.T.} \\ \text{with respect} \\ \text{to } x \end{pmatrix}$$

(y FIXED)

$$\frac{\partial}{\partial y} \hat{u} = \frac{\partial}{\partial y} \hat{u}(\xi, y)$$

$$\Delta u = u_{xx} + u_{yy} = 0$$

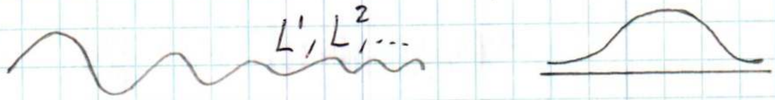
$\Delta \hat{u} = \hat{u}_{xx} + \hat{u}_{yy} = -\xi^2 \hat{u} + \hat{u}''_{(y)}$ if we "freeze" $\xi \in \mathbb{R}$,

this is an ODE in the variable y $\quad \hat{u}'' - \xi^2 \hat{u} = 0$ (linear homogeneous 2nd order const. coeff.)

general solution: $\hat{u} = a(\xi) e^{\xi y} + b(\xi) e^{-\xi y}$

(the arbitrary constants c_1 and c_2 here are two arbitrary functions of ξ)

$\hat{u}(\xi) \to 0$ as $\xi \to \pm\infty$ (because it is a F.T. ...)



[RIEMANN-LEBESGUE TH.]

$\Rightarrow a(\xi) = 0 \quad \xi > 0$ <u>and</u> $b(\xi) = 0$ for $\xi < 0$

(this way we cut away the parts of the exponentials that blow up to ∞)

<u>equivalently</u> $\quad \hat{u}(\xi, y) = \underbrace{C(\xi)}_{\hookrightarrow \text{ arbitrary function}} e^{-|\xi| y}$

Now the boundary condition <u>determines</u> uniquely

B.C. $\quad u(x,0) = f(x)$ and $\hat{u}(\xi, 0) = C(\xi)$

$\hat{u}(\xi, y) = \hat{f}(\xi) e^{-|\xi| y}$ (this the F.T. in x of the unique sol. of our full problem)

$$\Rightarrow u(x,y) = \mathcal{F}^{-1}\left(\hat{f}(\xi) e^{-|\xi| y}\right) =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} \hat{f}(\xi) e^{-y|\xi|}\, d\xi$$

(the inverse F.T. of a product is a convolution)

$$= \frac{1}{2\pi} f(x) * \frac{2y}{x^2 + y^2} \overset{(*)}{=} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(x-t)^2 + y^2} f(t)\, dt$$

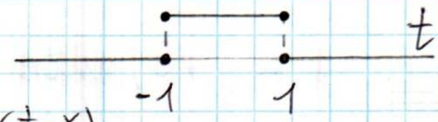
[CFR. 66 e 68R]

This is the solution.

In practice, if $f(x)$ is given explicitly, sometimes it will be convenient to compute the integral ($\star$). Sometimes it will be convenient to work with transforms and inverse transforms.

<u>Special case</u> $f(x) = \chi_{[-1,1]}(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{if } x \notin [-1,1] \end{cases}$

$$u(x,y) = \frac{y}{\pi}\int_{-1}^{1} \frac{dt}{(x-t)^2+y^2}$$

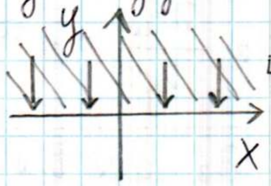


we multiply by $\frac{1/y^2}{1/y^2}$

$$= \frac{1}{\pi}\int_{-1}^{1} \frac{dt/y}{1+\left(\frac{t-x}{y}\right)^2} = \frac{1}{\pi}\int_{-1}^{1} \frac{d\left(\frac{t-x}{y}\right)}{1+\left(\frac{t-x}{y}\right)^2} =$$

$$= \frac{1}{\pi}\left\{ \operatorname{arctg}\frac{1-x}{y} - \operatorname{arctg}\frac{-1-x}{y} \right\} = \frac{1}{\pi}\left\{ \operatorname{arctg}\frac{1-x}{y} + \operatorname{arctg}\frac{1+x}{y} \right\}$$

at home given this $u(x,y)$ verify that it is harmonic in Ω (compute u_x, u_{xx}, u_y, u_{yy}, check $u_{xx}+u_{yy}=0$ for $(x,y) \in \Omega$)



$C^\infty(\Omega)$

The harmonic u (therefore also $C^\infty(\Omega)$) approximates the non-harmonic $f(x)$ at the boundary. "Regularization effect of ellyptic PDE's" or smoothing

<u>Note</u> if $y \to 0^+$

$$\lim_{y\to 0^+} u(x,y) = \begin{cases} 1 & \text{if } x \in (-1,1) \\ 0 & \text{if } |x| > 1 \\ 1/2 & \text{if } x = \pm 1 \end{cases}$$

$x \in (-1,1)$ $\quad \frac{1}{\pi}\{\underbrace{\operatorname{arctg}(\infty)}_{\pi/2} + \underbrace{\operatorname{arctg}(\infty)}_{\pi/2}\} \to 1$

$|x| > 1$ the two terms inside $\{\}$ cancel each other $\to 0$

if x is 1 or -1 <u>one</u> of the two terms inside $\{\}$ is 0

(MIDPOINT OF JUMP, LIKE IN F.S.) $\Longleftarrow$ $\frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$

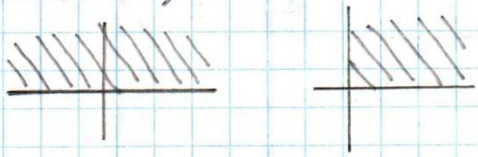
Linear PDE's of 2^{nd} order with constant coeff's

In 2 variables, the most general equation of this family can be written as:

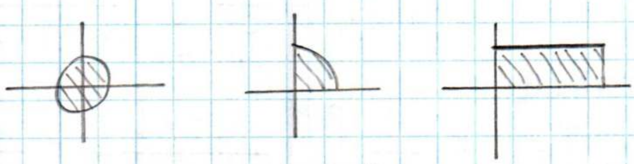
$$\boxed{a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + c\,u = f(x,y)}$$

for $(x,y) \in \Omega$ REGION (open, connected)

typical region we'll consider:



N.B. if $a_{12} \neq 0$, with an easy change of variables



we can get rid of the term with mixed partial derivatives:

$$x = \alpha\xi - \beta\eta \qquad\qquad \alpha = \cos\vartheta$$
$$y = \beta\xi + \alpha\eta \quad \text{where} \quad \beta = \sin\vartheta$$
$$\cot 2\vartheta = \frac{a_{11} - a_{22}}{2a_{12}}$$

We obtain (...) the PDE:

$$\boxed{\lambda_1\frac{\partial^2 u}{\partial x^2} + \lambda_2\frac{\partial^2 u}{\partial y^2} + 1^{st} \text{ order der.} = f(x,y)}$$

where λ_1 and λ_2 are the eigenvalues of the 2x2 matrix:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

Classification of linear, 2^{nd} order, const. coeff's PDE's:

(1) if λ_1 and λ_2 have the same sign, then our PDE is called ELLYPTIC (example just solved is ellyptic $\Delta = u_{xx} + u_{yy}$ $\lambda_1 = \lambda_2 = 1 > 0$)

(2) if either λ_1 or λ_2 is 0 (and the other $\neq 0$), then our PDE is PARABOLIC (ex. solved monday

$$\begin{pmatrix} -1 \text{ coeff of } u_{xx} \\ 0 \text{ coeff of } u_{yy} \end{pmatrix} \qquad \begin{matrix} u_t = u_{xx} \\ u_t - u_{xx} = 0 \end{matrix} \quad)$$

(3) if λ_1 and λ_2 have different signs, then our PDE is <u>HYPERBOLIC</u>. (see example next page)

Remarks: (1) This classification only uses the coeff's of the 2^{nd} order derivatives (the first order coeff's belong to the eq., BUT are irrelevant to the classification).

(2) <u>elliptic</u> and <u>parabolic</u> PDE's have a "smoothing effect". Hyperbolic DO NOT.

(3) this classification is based on <u>constant coeff's</u>. If the coeff's are not constant it might happen that the same PDE is elliptic in part of Ω and non-elliptic in an another part...

For example:
$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial t^2} = 0 \begin{cases} \text{elliptic if } x > 0 \\ \text{parabolic if } x = 0 \\ \text{hyperbolic if } x < 0 \end{cases}$$

<u>Tricomi equation</u>
(supersonic flight)

1st order PDE's $\quad u = u(x,y)$

General form: $\boxed{F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0}$

We will study (I) $a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = 0$ linear homogeneo 1st order

$\left\{\begin{array}{l}\text{The coeff's } a(x,y) \text{ and } b(x,y) \\ \text{maybe are constant, maybe not}\end{array}\right\}$ (Cauchy - Riemann is an example of this case) $a = 1/2; b =$

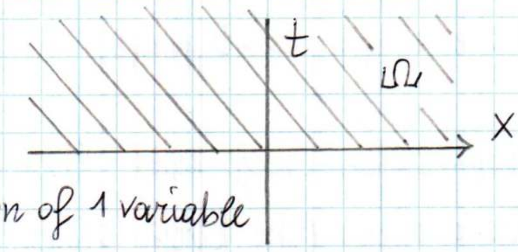
(II) $a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)\,u = d(x,y)$

N.B. (II) $\supset$ (I) linear PDE of 1st order

(III) $a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$

quasi-linear

Example of hyperbolic PDE (2^{nd} order) with suitable boundary data:



$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = u_0(x) \quad \text{given function of 1 variable} \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{given function of 1 variable} \end{cases}$$

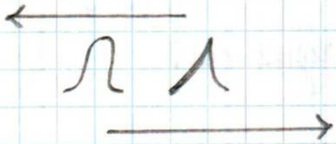
N.B. here we need 2 boundary conditions

Physical intuition: Vibrating string (idealized, ∞ lenght, where $u_0(x)$ is the initial displacement, $u_1(x)$ is the initial velocity and $t>0$ is time).

The general solution of the PDE has the form $\boxed{u(x,t) = F(x+t) + G(x-t)}$ where F, G, are arbitrary functions of 1 variable.

check:
$$u_t = F'(x+t)\cdot 1 + G'(x-t)(-1) = F'(x+t) - G'(x-t)$$
$$u_{tt} = F''(x+t) - G''(x-t)(-1) = F''(x+t) + G''(x-t)$$
$$u_x = F'(x+t) + G'(x-t)$$
$$u_{xx} = F''(x+t) + G''(x-t)$$
therefore $u_{xx} - u_{yy} = 0$

Obs. F, G are completely arbitrary BUT they must have two derivatives (maybe 2^{nd} derivatives have discontinuities but they must exist)



Now, let's apply the boundary conditions

$$u(x,t) = f(x+t) + g(x-t)$$

$$\begin{cases} f(x) + g(x) = u_0(x) \quad \text{given} \\ f'(x) - g'(x) = u_1(x) \quad \text{given} \Rightarrow \underset{\text{F.T.C.}}{f(x) - g(x) = \int_0^x u_1(\xi)\, d\xi + c} \end{cases}$$

$$\Rightarrow 2f(x) = u_0(x) + \int_0^x u_1(\xi)\, d\xi + c \qquad \text{(summation)}$$

$$2g(x) = u_0(x) - \int_0^x u_1(\xi)\, d\xi - c \qquad \text{(subtraction)}$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\, d\xi}$$

Physically we have two waves with the initial shape of $u_0(x)$ overlapped with another wave that depends on the initial velocity (NO FRICTION in this model).

LESS.12 08/01/09

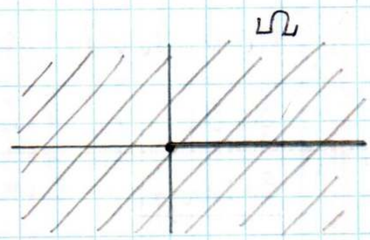
EX: 1st order PDE: find $u \equiv u(x,y)$ for $(x,y) \in \mathbb{R}^2 \equiv \Omega$ satisfying $\partial u / \partial y = 0$ in Ω

Suppose $A(t)$ is an arbitrary function of 1 variable, then $u(x,y) = A(x)$ is the general solution. In this case $A(x)$ does not even need to be differentiable or continuous!

Modified versions of the same example:

$$\frac{\partial u}{\partial y} = 0 \text{ in } \Omega \equiv \mathbb{R}^2 \setminus \text{Positive real axis}$$

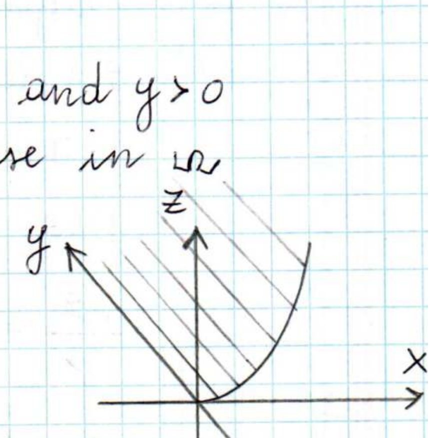
$$\equiv \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} \text{not of the} \\ \text{form } (x,0) \text{ with } x \geq 0 \end{array} \right\}$$



In this case $u(x,y) = A(x)$ is still a solution, but there are other solutions:

$$\text{e.g. } u(x,y) = \begin{cases} x^2 & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise in } \Omega \end{cases}$$

This case shows that even if we have all the solutions of a PDE in Ω, it is not true that a restriction of these solutions to $\Omega' \subsetneq \Omega$ provide all the solutions of the same PDE in Ω'.

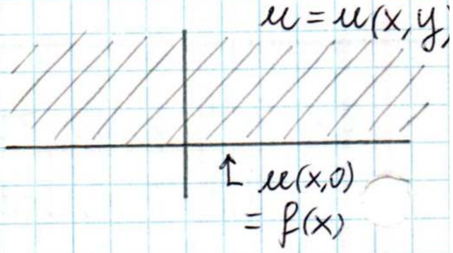


[P.120]

N.B. the same observation holds for PDE's of 2nd order (or higher).

A <u>well posed</u> PDE problem has the following properties:

1) we have the PDE itself and a <u>domain</u> Ω where we want $u = u(x,y)$ to satisfy it.
2) u and/or its partial derivatives must satisfy a <u>boundary condition</u> on $\partial\Omega$.
3) the solution to (1)(2) must be <u>unique</u>.
4) "small" perturbations of the boundary data produce "small" perturbations in the solution.
(e.g. if $u(x,0) = f(x)$ given function
suppose $\|f\|_{L^2(\mathbb{R})} = 3$ and $\|u\|_{L^2(\Omega)} = 12$



if we change f slightly to $f(x) + \varepsilon(x)$ with $\|f+\varepsilon\|_2 = 3{,}01$
then $\|u\|_{L^2(\Omega)}$ cannot be very far from 12).

5) there are notions of <u>stability</u> of a solution measuring how much u changes when we change the boundary data on $\partial\Omega$.

<u>example</u>: find 1st order PDE satisfied by all planes $u = u(x,y)$ that intersect the z-axis at the origin. (means $c=0$)

$z = ax + by$ $\quad a, b$ parameters

$ax + by - z = 0$ the vector $(a, b, -1)$ is $\perp$ to the plane

(N.B. same thing for $\alpha(a,b,-1) \equiv (\alpha a, \alpha b, -\alpha)$)

$$\frac{\partial z}{\partial x} = a \qquad \frac{\partial z}{\partial y} = b$$

$$z = \frac{\partial z}{\partial x} x + \frac{\partial z}{\partial y} y$$

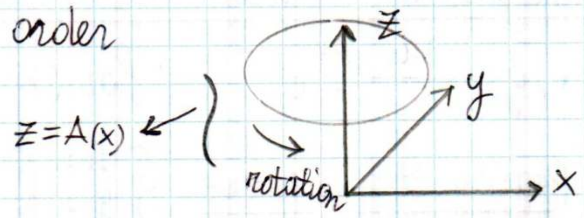
$$\boxed{x u_x + y u_y - u = 0}$$ linear, 1st order, non-constant coeff's.

In principle (even though the computations might be more complicated than the previous example) we can apply the same procedure to any <u>family of surfaces</u> $z \underset{*}{=} f(x,y,a,b)$

$$\frac{\partial z}{\partial x} \underset{**}{=} f_x(x,y,a,b) \qquad \frac{\partial z}{\partial y} \underset{***}{=} f_y(x,y,a,b)$$

eliminating the parameters a, b from the 3 equalities $*, **, ***$ we can obtain a PDE of 1st order.

Let us show that all <u>surfaces of rotation</u> are solutions of a PDE of 1st order



Suppose $A(x)$ is an arbitrary function of 1 variable $x \in (a,b)$ $\boxed{b > a \geq 0}$ if we set $r = \sqrt{x^2+y^2}$, then $z = A(r) =$
$= A(\sqrt{x^2+y^2})$ is the equation of our surface.

$$\frac{\partial z}{\partial x} = A'(\sqrt{x^2+y^2}) \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = A'(r)\frac{x}{r}$$

$$\frac{\partial z}{\partial y} = A'(\sqrt{x^2+y^2}) \frac{y}{\sqrt{x^2+y^2}} = A'(r)\frac{y}{r}$$

$$\Rightarrow A'(r) = \frac{\cancel{r}}{x} z_x = \frac{\cancel{r}}{y} z_y$$

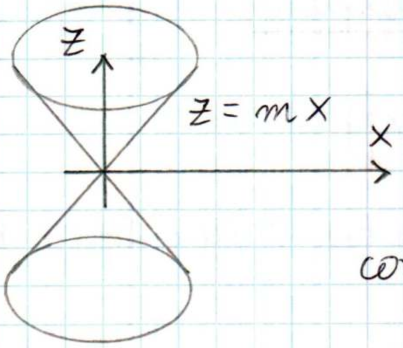
$$\Rightarrow \frac{1}{x} z_x = \frac{1}{y} z_y \qquad y z_x = x z_y$$

$\boxed{y z_x - x z_y = 0}$ linear, homogeneous PDE 1st order with non-constant coeff's

its solutions in $\Omega \equiv \mathbb{R}^2$ are <u>all</u> the surfaces of rotation around the z axis.

<u>N.B.</u> for this kind of PDE in $\mathbb{R}^2$ the usual boundary condition is not suitable but it makes sense to solve a Cauchy Problem.

For example, associate with $yz_x - xz_y = 0$ the requirement that our surface goes through the line $\ell : \begin{cases} x = t \\ y = 0 \\ z = mt \end{cases}$



Given a PDE, the problem of finding a solution surface that contains a given curve is called a Cauchy p

---

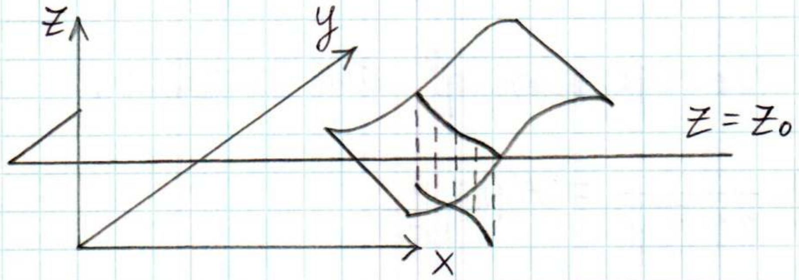
Let us look at <u>homogeneous linear PDE's of 1st order</u>

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \overset{(*)}{=} 0 \qquad \text{for } (x,y) \in \Omega$$

Suppose that the <u>(parametric) curve</u> $\begin{cases} x = f(t) \\ y = g(t) \\ z = z_0 \end{cases}$ $t \in (c,d)$

is a <u>level curve</u> on a solution surface of (*)

[P.124]



such that $f, g \in C'((c,d))$ => the composite function

$u(f(t), g(t)) = z_0 \;\; \forall t \in (c,d)$

$\frac{d}{dt} u(f(t), g(t)) = 0$ for $t \in (c,d)$

$$\Rightarrow \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = 0$$

that shows that there is a close connection between the solution surfaces of (*) and the solutions of

$$\boxed{\begin{cases} dx/dt = a(x,y) \\ dy/dt = b(x,y) \end{cases}}$$

System of 2 ODE's of 1st order (called <u>CHARACTERISTIC EQUATIONS</u>)

<u>Theorem</u> Suppose $\Omega \subset \mathbb{R}^2$ is a <u>region</u> and $a(x,y)$, $b(x,y)$ are functions in $C^1(\Omega)$. Suppose $P_0 \equiv (x_0, y_0, z_0)$ is a point in $\mathbb{R}^3$ such that its projection (x_0, y_0) belongs to Ω; <u>then</u> $\exists!$ a characteristic curve of the homogeneous PDE

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \overset{(*)}{=} 0 \quad (x,y) \in \Omega$$

<u>Def.</u> each curve solution of $\begin{cases} dx/dt = a(x,y) \\ dy/dt = b(x,y) \end{cases}$ $\quad (x,y) \in \Omega$

is called a <u>characteristic curve</u> of $(*)$.

<u>N.B.</u> the restriction of a characteristic curve to $\Omega' \subsetneq \Omega$ maybe/maybe not a characteristic curve for the smaller region Ω' (in general, we need to solve the system for the smaller region).

<u>Theorem:</u> suppose $a(x,y)$, $b(x,y) \in C^1(\Omega)$; <u>then</u> $u \equiv u(x,y)$ is a solution of the PDE $a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \overset{(*)}{=} 0 \iff$ the surface $z = u(x,y)$ is a union of characteristic curves with $u \in C^1(\Omega)$.

<u>Theorem</u>: if $u = u(x,y)$ is a surface of $(*)$ and $A(t)$ is an arbitrary $C^1((c,d))$ function, then $A(u(x,y))$ <u>is also</u> a solution

<u>Proof</u> we need to check that if $u(x,y)$ satisfies PDE $(*)$ also $A(u(x,y))$ satisfies the same PDE. Let's plug $A(u(x,y))$ in the PDE:

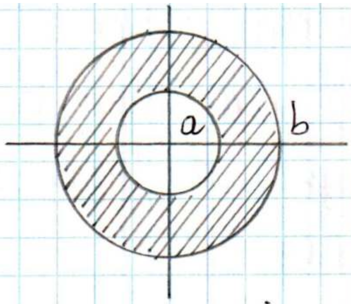
$$a(x,y)\frac{\partial}{\partial x}A(u(x,y)) + b(x,y)\frac{\partial}{\partial y}A(u(x,y)) = a(x,y)A'(u(x,y))\frac{\partial u}{\partial x} + b(x,y)A'(u(x,y))\frac{\partial u}{\partial y}$$

$$= A'(u)\left[a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y}\right] = 0$$

Because, by assumption, the quantity in [...] is $=0$

Q.E.D.

Ω
anulus



$\Omega \equiv \{(x,y) \in \mathbb{R}^2 : a^2 < x^2 + y^2 < b^2\}$

Remark 1: this region Ω is connected but is not simply connected.

Remark 2: $\partial\Omega$ (the boundary of Ω) is the union of the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ centrated at $(0;0)$

In this region we want to solve the PDE $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0$ $(\bigstar)$
using the method of characteristics.
(we have seen before that its solutions are surfaces of rotation)

N.B. $(\bigstar)$ is an homogeneous linear PDE with $a(x,y) = y$
$b(x,y) = -x$

N.B. the characteristic equations associated with $(\bigstar)$ are

$$\begin{cases} dx/dt = y(t) \\ dy/dt = -x(t) \end{cases}$$

(System of 2 ODE's of 1st order, equivalent to one ODE 2nd order)

$$\begin{cases} x'(t) = y(t) \Rightarrow x''(t) = y'(t) \\ y'(t) = -x(t) \end{cases}$$

$\Rightarrow x''(t) = -x(t)$ $\quad x''+x=0$

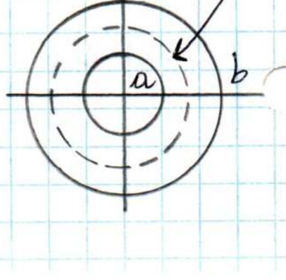
$\alpha^2+1=0 \quad \alpha^2=-1 \quad \alpha \begin{cases} +i \\ -i \end{cases}$

$x = c_1 e^{it} + c_2 e^{-it} = d_1 \cos t + d_2 \sin t$

$y(t) = x'(t) = -d_1 \sin t + d_2 \cos t$

Another way to write $x(t)$ and $y(t)$ is $x = r\cos(t-\alpha)$, $y = -r\sin(t$
In order for this (circle) curve to be in Ω it must satisfy $a < r < b$

The characteristic curves of $(\bigstar)$ are circles centered at $(0,0)$ of radius greater than a, smaller than b.



Every function $z = A(x^2+y^2)$ is constant on each characteristic curve so it's a solution of (✡) (surface of rotation).
N.B. before we had $A(\sqrt{x^2+y^2})$ but this is the same...

Linear, non-homogeneous PDE's of 1st order:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \overset{**}{=} c(x,y)$$

<u>Theorem</u> Suppose $u_p(x,y)$ is a surface solution of ** (the full equation); if we add to this $u_p(x,y)$ all the solutions of the associated homogeneous eq. $a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = 0$,

then we obtain all the solutions of ** (analogy with linear ODE

<u>Proof</u> Suppose $u_1(x,y)$ is a solution of **. Consider the operator
$L \underset{def.}{=} a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$ which is <u>linear</u> (easy to check)

Compute $L(u_1 - u_p) = L(u_1) - L(u_p) = c(x,y) - c(x,y) = 0$

$\Rightarrow u_1 - u_p$ is a solution of $Lu = 0$

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \overset{*}{=} 0 \qquad Lu \overset{*}{=} 0$$

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \overset{**}{=} c(x,y) \quad Lu \overset{**}{=} c(x,y)$$

therefore any solution of ** can be obtained adding $u_p(x,y)$ to a certain solution of * $[\, u_1 = \underbrace{u_p}_{\substack{\text{solution} \\ \text{of } **}} + \underbrace{(u_1 - u_p)}_{\substack{\text{solution} \\ \text{of } *}} \,]$

Vice-versa, suppose $u_2(x,y)$ is in the general solution of *
$Lu = 0$ then $L(u_2 + u_p)$ (with, as before, u_p solution of **) $=$
$= L(u_p) + L(u_2) = c(x,y) + 0 = c(x,y)$ Q.E.D.

Quasi-linear PDE's of 1^{st} order:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \qquad (x,y) \in \Omega$$

N.B. our unknown function $u \equiv u(x,y)$ can appear non-linear in the coeff's $a(x,y,u)$ and $b(x,y,u)$, BUT the partial derivatives $u_x(x,y)$ and $u_y(x,y)$ appear linearly in the PDE.

characteristic equations:

$$\begin{cases} dx/dt = a(x,y,z) \\ dy/dt = b(x,y,z) \\ dz/dt = c(x,y,z) \end{cases}$$

system of 3 ODE's with unknown functions $x(t), y(t), z(t)$.

Suppose that we can find $x = f(t)$, $y = g(t)$, $z = h(t)$, $t \in (c,d)$ solution of this system. These are called characteristic curves.

Remark in the previous case the charact. curves were // to the xy plan here NOT.

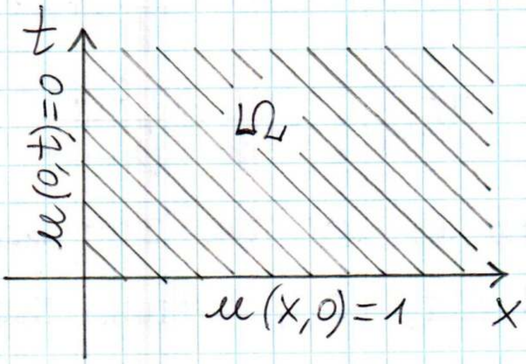
Theorem suppose $a(x,y,z)$, $b(x,y,z)$, $c(x,y,z)$ are $C^1(\Omega \times \mathbb{R})$ then $u(x,y)$ is a solution of the quasi-linear eq. with coeff's $a, b, c \iff u(x,y) \in C^1(\Omega)$ and the surface $z = u(x,y)$ is a union of characteristic curves.

Remark to solve boundary problems for PDE's of 2^{nd} order (elliptic, parabolic, hyperbolic) we mostly use: Fourier Transform, Laplace Transform, Fourier Series (or generalized F.S., including orthogonal polynomials, Bessel functions).

The approach used in the book for 1^{st} order PDE's does not mention the use of transforms. But sometimes it is in fact useful to use transforms (or F.S. ...) also for 1^{st} order PDE. (especially so if we have a boundary problem instead of a Cauchy Problem).

<u>Example</u>: application of the Laplace Transform to a 1^{st} order PDE:

$$\begin{cases} \dfrac{1}{x+1} u_t + u_x = 0 & \text{in } \Omega \\ u(x,0) = 1 & \text{on } \partial\Omega \\ u(0,t) = 0 & \text{on } \partial\Omega \end{cases}$$



$$\Omega \equiv \left\{ (x,t) \in \mathbb{R}^2 : x > 0,\ t > 0 \right\}$$

$$\partial\Omega \equiv \left\{ (x,0) \in \mathbb{R}^2 : x \geq 0 \right\} \cup \left\{ (0,t) \in \mathbb{R}^2 : t \geq 0 \right\}$$

Laplace transform of $u(x,t)$ w.r.t. t ("partial" L.T.)

$$v(x,s) = \int_0^{+\infty} u(x,t)\, e^{-st}\, dt \qquad (x \text{ FIXED})$$

PDE: $u_x = -\dfrac{1}{x+1} u_t \qquad v_x = -\dfrac{1}{x+1} \mathcal{L}\{u_t\}$

$$v_x = -\frac{1}{x+1} [\, s\, v(x,s) - \underbrace{u(x,0)}_{\substack{\parallel \\ 1}} \,]$$

because of the 1^{st} boundary condition

$$v_x(x,s) + \frac{s}{x+1} v(x,s) = \frac{1}{x+1} \qquad \text{(linear, first order ODE)}$$

$$\int \frac{s}{x+1}\, dx = s \log(x+1) = \log(x+1)^s$$

integral factor $e^{\log(x+1)^s} = (x+1)^s$

We multiply both members of the equation by $(x+1)^s$:

$$\underbrace{(x+1)^s v_x + s(x+1)^{s-1} v}_{} = (x+1)^{s-1}$$

Now we recognize this term to be $\dfrac{d}{dx}[(x+1)^s v]$

$$\frac{d}{dx}[(x+1)^s v] = (x+1)^{s-1}$$

The integral is:

$$(x+1)^s\, V = \frac{1}{s}(x+1)^s + C(s) \qquad \left(\begin{array}{l}\text{the arbitrary constant}\\ \text{here is a function of } s\end{array}\right)$$

$$V_{(x,s)} = \frac{1}{s} + \frac{C(s)}{(x+1)^s}$$

Now we use the 2^{nd} boundary condition:

$u(0,t)=0$, which transform is $V(0,s)=0$, so:

$$0 = V_{(0,s)} = \frac{1}{s} + \frac{C(s)}{1} \implies C(s) = -\frac{1}{s}$$

Therefore:

$$V(x,s) = \frac{1}{s} - \frac{1}{s(x+1)^s}$$

Let's try some manipulation in order to apply L.T. properties from tables:

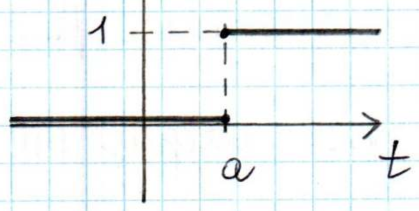
$$V_{(x,s)} = \frac{1}{s} - \frac{1}{s} e^{-s \overbrace{\log(x+1)}^{\text{THIS IS A CONSTANT}}} \qquad \left(\frac{1}{(x+1)^s} = e^{\log(x+1)^{-s}} = e^{-s\log(x+1)}\right)$$

$$u_{(x,t)} = \mathcal{L}^{-1}\left[V(x,s)\right] = 1 - H\big(t - \log(x+1)\big)$$

Actually:

$$f(t) = 1 = H(t) \qquad \mathcal{L}(H(t)) = \frac{1}{s}$$

$$g_{(t)} = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \end{cases} = H(t-a)$$



$(a>0)$

$$\mathcal{L}[g(t)] = \int_a^{+\infty} 1 \cdot e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_a^{+\infty} = \frac{1}{s} e^{-sa}$$

Note that:

if $t > \log(x+1)$: $\quad u(x,t) = 1-1 = 0$

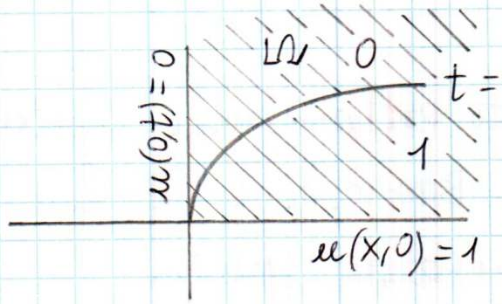
if $t < \log(x+1)$: $\quad u(x,t) = 1-0 = 1$

Viceversa, sometimes (not very often) it is useful to extend the method of characteristics to PDE's of 2nd order (we'll not do this in this course).

ex. find $u \equiv u(x,t)$ sol. of $\begin{cases} \frac{1}{x+1} u_t + u_x \overset{(*)}{=} 0 \\ u(x,0) = 1 \\ u(0,t) = 0 \end{cases}$ $x>0$, $t>0$

$\Omega \equiv \{(x,t) \in \mathbb{R}^2 : x>0, t>0\}$

$\partial\Omega \equiv \{(x,0) \in \mathbb{R}^2 : x \geq 0\} \cup \{(0,t) \in \mathbb{R}^2 : t \geq 0\}$ union of 2 half-lines



By the way $(*)$ is a 1st order linear homogeneous PDE. Using Laplace Transform we can show that the unique solution (with boundary conditions) is $u(x,t) = 1 - H(t - \log(x+1))$. $H(t)$

By the theory seen today we know that $A(1-H(t-\log(x+1)))$ is also a solution of $\frac{1}{x+1} u_t + u_x = 0$, but it does not satisfy the boundary conditions when $A(x) = x$.

Wavelets (special types of orthogonal systems in $L^2(\mathbb{R})$)

Historical note: similar to classical Fourier analysis, but more modern.

F.S./F.T. end of 1700's, beginning 1800's.

Haar Basis (easiest wavelet)

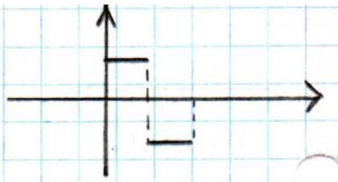
1930-1940 Littlewood-Paley theories.

Possible applications: 1) solution of PDE's
2) compression of signals (sound, pictures, movies)

Haar Basis

Def. $\psi(x) = \begin{cases} 1 & \text{if } x \in (0; \frac{1}{2}) \\ -1 & \text{if } x \in (\frac{1}{2}; 1) \\ 0 & \text{if } x \notin (0; 1) \end{cases}$

MOTHER FUNCTION



Def. $\boxed{\psi_{j\kappa}(x) = 2^{j/2} \cdot \psi(2^j x - \kappa)}$ for $\kappa \in \mathbb{Z}$ $j \in \mathbb{Z}$

$\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, \dots\}$

N.B. $\psi_{j\kappa}(X)$ are dyadic dilations and integer translations of one "mother" function.

Theorem (Haar) $\psi_{j\kappa}(X)$ is complete ortho-normal system in $L^2(\mathbb{R})$. It's also a Schauder Basis in $L^p(\mathbb{R})$ for $1 < p < \infty$ and in many other function spaces.

We can compute the wavelet coeff's (generalized Fourier):

$$c_{j\kappa} \overset{\text{def.}}{=} (\psi_{j\kappa}; f) = \int_{-\infty}^{+\infty} f(x)\, \psi_{j\kappa}(x)\, dx =$$

$$\left(\begin{array}{c} \|\psi_{j\kappa}\|_2^2 = 1 \\ \text{because orthonormal} \end{array} \right) = 2^{j/2} \int_{-\infty}^{+\infty} f(x)\, \psi(2^j x - \kappa)\, dX$$

$$f(x) \sim \sum_{j,\kappa \in \mathbb{Z}} c_{j\kappa} \psi_{j\kappa}(x)$$

More complicated "mother" functions $\psi(x)$ can be chosen to have better properties of $c_{j\kappa}$ w.r.t. derivatives...
The structure is the same: dilate dyadically and traslate of an integer amount a given $\psi(x)$ mother function.
Compare with classical Fourier Series: the mother function is $\sin X$
Only 2 translations (sin/cos). Dilations are not dyadic but integer $\mathbb{N}$.