# Elastic Theory of Plates

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**Abstract**

In this memoir the elastic theory of plates will be reviewed following different authors, amongst the others Timoshenko and Woinowsky-Krieger (1959), Belluzzi (1966), Selvadurai (1979), Ventsel and Krauthammer (2001), Corigliano and Taliercio (2005). Along this memoir, the elastic theory of plates is explained starting from the general theory, passing through rectangular plates and finishing with the theory of thin plates. All this process describes how to derive the elastic equations for circular thin plates. These equations are achieved via a transformation of the reference system from rectangular to polar coordinates. The axial symmetry of circular plates simplifies the problem to one spatial variable $r$, thus making the dynamic analysis more manageable.

## 1 The plate model

Let us start off by considering a generic plate element as shown in figure 1. The plate model can be viewed as a two-dimensional extension of the beam model. The basic idea is to analyse the plate deformation by studying the deformation of its middle plane. In this way, the state of deformation will be associated to the loads acting in the middle plane of the plate. As in the beam model the beam deformation is analyse by studying its axis, analogously herein the plate deformation is analysed by referring to its middle plane.
The displacement in the vertical direction $z$ is defined as $w \equiv w(x, y)$, i.e., it is function of $x$ and $y$, but not of $z$.
The hypotheses made in order to develop the plate model are the following ones:

- small displacements and small deformations;

- homogeneous, isotropic, Green hyper-elastic material (i.e. there exists a potential function by which stresses and strains can be represented);

- the medium is a Cauchy continuum (that is, the stress-state tensor is symmetric, and there are no distributed micro-couples);
- two geometrical dimensions are prevalent with respect to the third one;
- $\sigma_z = 0$, hypothesis that does not allow to represent the state of stress diffusivity.

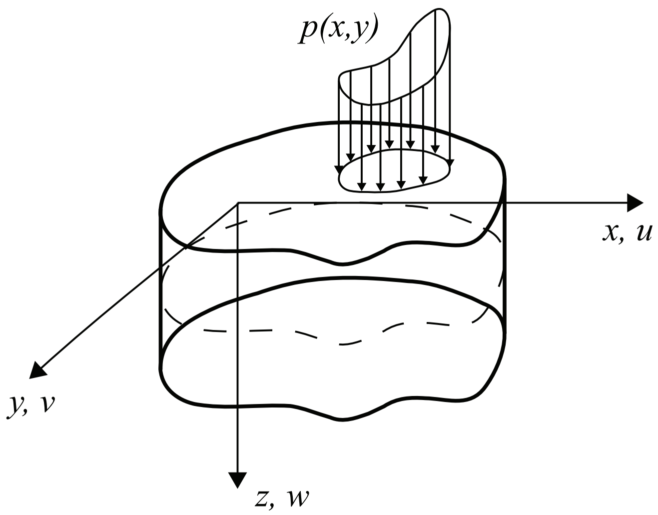


Figure 1: A generic plate element with the reference system lying on its middle plane.

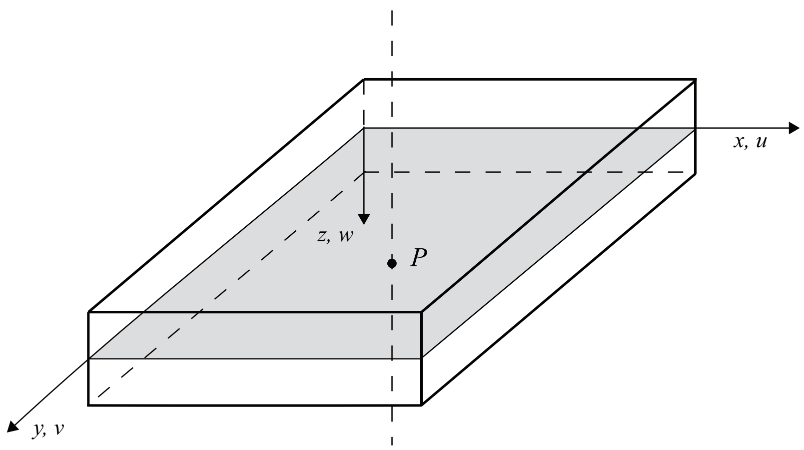


Figure 2: A generic point on the generic straight segment initially orthogonal to the plate middle plane.

The kinematic model of the deflected plate assumes that a generic straight segment, initially perpendicular to the middle plane (see figure 2), after the deformation it is still straight. Not necessarily, after the deformation, the generic straight segment is still perpendicular to the deformed mid plane, as shown in figure 3.

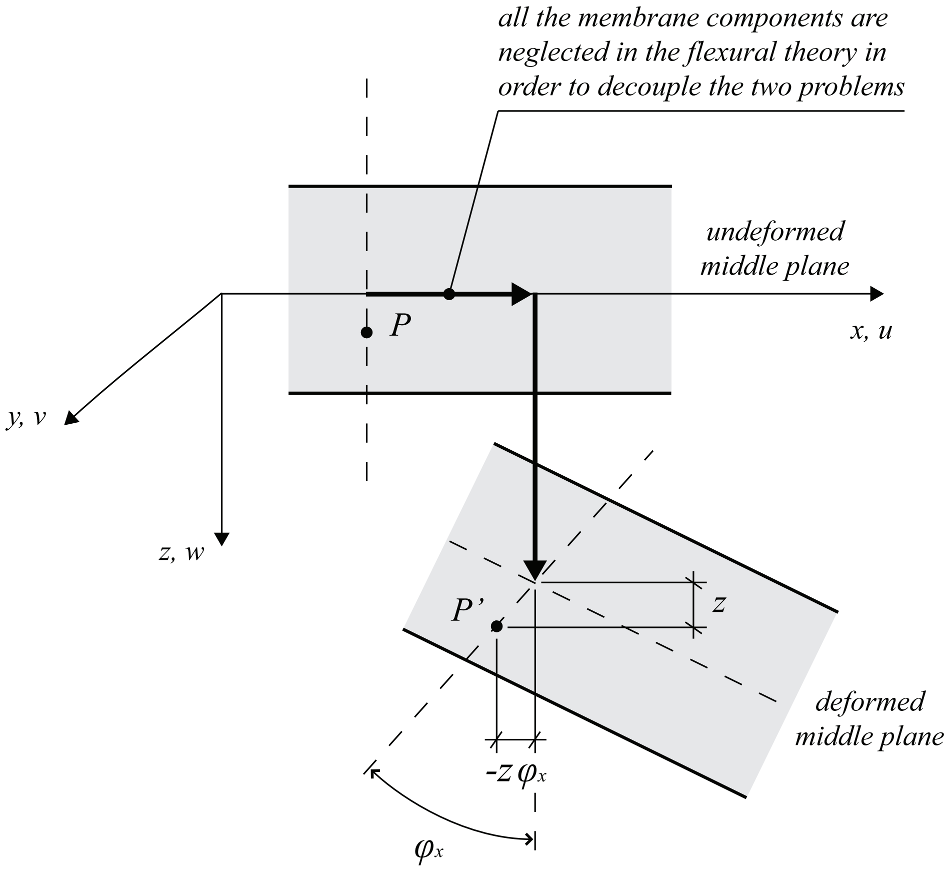


Figure 3: A section of a plate, traced in the $x$-$z$ plane, before and after the deformation.

Within this discussion the focus will be on the flexural behaviour of plates, thus only forces acting perpendicularly to the middle plane will be considered, decoupling the flexural problem from the one related to the forces acting parallel to the middle plane (membrane theory).

**Displacement components.** The local displacement vector is represented by:

$$\underline{s}(x,y,z) = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -z\varphi_x(x,y) \\ -z\varphi_y(x,y) \\ w(x,y) \end{bmatrix},$$

where:

- $u$ is the displacement component in the $x$ direction;
- $v$ is the displacement component in the $y$ direction;
- $w$ is the displacement component in the $z$ direction.

and $\varphi_x(x,y)$, $\varphi_y(x,y)$ and $w(x,y)$ are the generalized displacements:

- $\varphi_x(x, y)$ is the rotation around the $y$ axis occurring in the $x$-$z$ plane;
- $\varphi_y(x, y)$ is the rotation around the $x$ axis occurring in the $y$-$z$ plane;
- $w(x, y)$ is the middle plane displacement in the vertical direction $z$.

The displacement vector can be rewritten as:

$$\underline{s} = \underline{\underline{n}}\,\underline{U},$$

where $\underline{U}$ is the vector of generalized displacements:

$$\underline{U} = \begin{bmatrix} w(x,y) \\ \varphi_x(x,y) \\ \varphi_y(x,y) \end{bmatrix},$$

and $\underline{\underline{n}}$ is the correlation matrix between local displacements and generalized ones:

$$\underline{\underline{n}} = \begin{bmatrix} 0 & -z & 0 \\ 0 & 0 & -z \\ 1 & 0 & 0 \end{bmatrix}.$$

**Strain components.** The strain components can be worked out by means of the compatibility equations:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y} + v_{,x} \\ u_{,z} + w_{,x} \\ w_{,y} + v_{,z} \end{bmatrix} = \begin{bmatrix} -z\varphi_{x,x} \\ -z\varphi_{y,y} \\ 0 \\ -z\varphi_{x,y} - z\varphi_{y,x} \\ -\varphi_x + w_{,x} \\ -\varphi_y + w_{,y} \end{bmatrix} = \underline{\underline{b}}\,\underline{q},$$

where $\underline{q}$ is the vector of generalized strains and $\underline{\underline{b}}$ is the correlation matrix between local strains and generalized ones.

$$\underline{q} = \begin{bmatrix} -\varphi_{x,x} \\ -\varphi_{y,y} \\ -(\varphi_{x,y} + \varphi_{y,x}) \\ -\varphi_x + w_{,x} \\ -\varphi_y + w_{,y} \end{bmatrix} = \begin{bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ t_x \\ t_y \end{bmatrix}. \tag{1}$$

The terms denoted with $\chi$ are the generalized curvatures; in particular, $\chi_{xy}$ is the torsional curvature. The terms $t_x$ and $t_y$ represent the shear angular deformations.

$$\underline{\underline{b}} = \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

**Load components.** The generalized loads will be worked out by using the definition of external specific work per unit area.

$$\underline{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

The external specific work per unit area is given by:

$$\frac{\mathrm{d}W_E}{\mathrm{d}A} = \int_{-h/2}^{+h/2} F_i \delta\hat{s}_i \,\mathrm{d}z = \underline{P}^T \delta\underline{\hat{U}} = \int_{-h/2}^{+h/2} \delta\underline{\hat{s}}^T \underline{F} \,\mathrm{d}z,$$

where:

- $\delta\hat{s}_i$ is the virtual displacement field;
- $\underline{P}$ is the vector of the generalized loads.

$$\frac{\mathrm{d}W_E}{\mathrm{d}A} = \delta\underline{\hat{U}}^T \int_{-h/2}^{+h/2} \underline{\underline{n}}^T \underline{F} \,\mathrm{d}z = \underline{P}^T \delta\underline{\hat{U}} = \delta\underline{\hat{U}}^T \underline{P}.$$

From the last equation one can read the expression that give rise to the generalized loads:

$$\underline{P} = \int_{-h/2}^{+h/2} \underline{\underline{n}}^T \underline{F} \,\mathrm{d}z.$$

Substituting the expressions for $\underline{\underline{n}}$ and $\underline{F}$ one can get:

$$\begin{aligned}
\underline{P} &= \int_{-h/2}^{+h/2} \begin{bmatrix} 0 & 0 & 1 \\ -z & 0 & 0 \\ 0 & -z & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \mathrm{d}z, \\
&= \int_{-h/2}^{+h/2} \begin{bmatrix} F_z \\ -zF_x \\ -zF_y \end{bmatrix} \mathrm{d}z, \\
&= \begin{bmatrix} p(x,y) \\ m_x(x,y) \\ m_y(x,y) \end{bmatrix}.
\end{aligned}$$

Note that $p(x, y)$ is dimensionally a force per unit area (i.e., a surface distributed load, [F/L$^2$]), whilst $m_x(x, y)$ and $m_y(x, y)$ are moments per unit length (i.e., they have the dimension of a force, [F]).
It should be noted that there is no explicit information about the points where the generalized loads are acting; it is only the assumption made by the model that permits to tell that they act in the middle plane of the plate, as shown in figure 4.

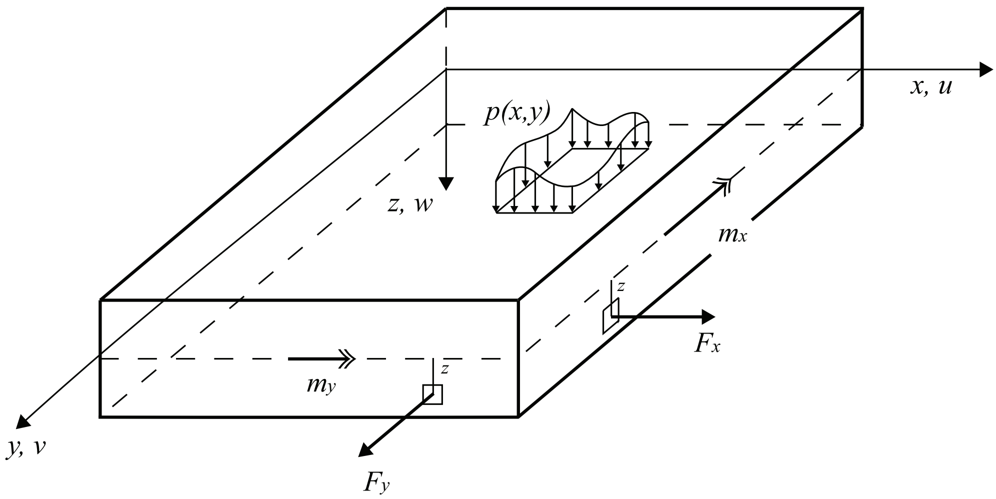


Figure 4: Generalized loads acting on a rectangular plate element.

**Stress components.** In order to work out the generalized stresses, the definition of internal specific work per unit area will be exploited:

$$\frac{\mathrm{d}W_I}{\mathrm{d}A} = \int_{-h/2}^{+h/2} \delta\hat{\underline{\varepsilon}}^T \underline{\sigma}\,\mathrm{d}z = \delta\underline{q}^T \int_{-h/2}^{+h/2} \underline{\underline{b}}\underline{\sigma}\,\mathrm{d}z = \delta\underline{q}^T \underline{Q}, \tag{2}$$

where:

- $\delta\hat{\underline{\varepsilon}}$ are the virtual local strains;
- $\underline{\sigma}$ are the local stresses;
- $\underline{Q}$ is the vector containing the generalized stresses.

The local deformations are related to the generalized ones by means of the correlation matrix $\underline{\underline{b}}$:

$$\underline{\varepsilon} = \underline{\underline{b}}\underline{q},$$

The expression needed in order to work out the vector of the generalized stresses can be easily read from equation (2):

$$\underline{Q} = \int_{-h/2}^{+h/2} \underline{\underline{b}}\underline{\sigma}\,\mathrm{d}z.$$

Performing the computations one can finally obtain:

$$\underline{Q} = \int_{-h/2}^{+h/2} \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \mathrm{d}z,$$

$$= \begin{bmatrix} \int_{-h/2}^{+h/2} z\sigma_x \,\mathrm{d}z \\ \int_{-h/2}^{+h/2} z\sigma_y \,\mathrm{d}z \\ \int_{-h/2}^{+h/2} z\tau_{xy} \,\mathrm{d}z \\ \int_{-h/2}^{+h/2} \tau_{xz} \,\mathrm{d}z \\ \int_{-h/2}^{+h/2} \tau_{yz} \,\mathrm{d}z \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \\ V_x \\ V_y \end{bmatrix},$$

where $\sigma_x$ and $\sigma_y$ are the normal stresses, whilst $\tau_{xy}$, $\tau_{xz}$ and $\tau_{yz}$ are the tangential stresses. The distance of the point of application of such stresses from the middle plane is denoted as $z$, as can be seen in figure 5.
The generalized moments $M_x$, $M_y$ and $M_{xy}$ have the dimension of a force (i.e., they are moments per unit length, [F]), whilst the shear terms $V_x$ and $V_y$ have the dimensions of a force per unit length, i.e. [F/L].
All the local and generalized stresses, along with the directions in which they are acting, are graphically illustrated in figure 5. The moments are represented as vectors; $z$ represents the stresses lever arm with respect to the middle plane.

# 2 Plate equilibrium problem

There are three different ways to study the problem of the plate equilibrium:

- by using the virtual work principle;
- by using the integrated equilibrium equations;
- by studying the equilibrium of a plate element.

In this section, the problem of the plate equilibrium will be studied by means of a rectangular plate element, as illustrated in figure 5.

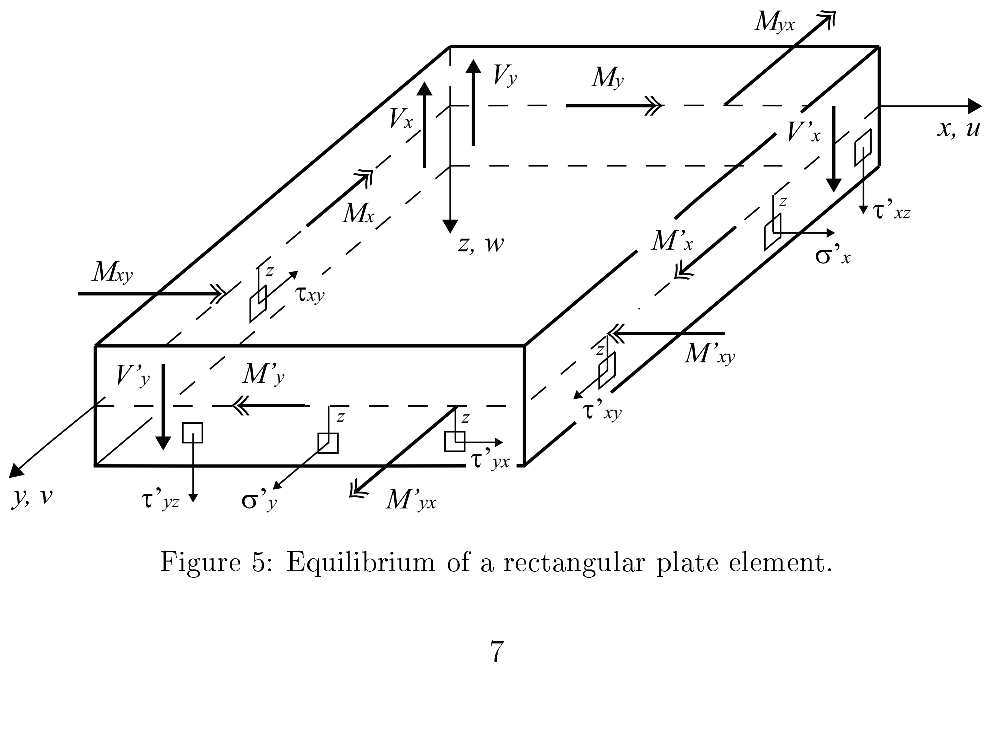


Figure 5: Equilibrium of a rectangular plate element.

**Rotational equilibrium with respect to $x$ axis.**

$V_y' \,\mathrm{d}x \,\mathrm{d}y - M_y' \,\mathrm{d}x + M_y \,\mathrm{d}x + M_{xy}' \,\mathrm{d}y + M_{xy} \,\mathrm{d}y + p(x,y) \,\mathrm{d}x \,\mathrm{d}y \,\frac{\mathrm{d}y}{2} + m_y \,\mathrm{d}x \,\mathrm{d}y = 0.$

The term $p(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}y}{2}$ is dropped out since it represents an infinitesimal of higher order.

$$\left(V_y + \frac{\partial V_y}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}y - \left(M_y + \frac{\partial M_y}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x + M_y\,\mathrm{d}x \\ - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y + M_{xy}\,\mathrm{d}y + m_y\,\mathrm{d}x\,\mathrm{d}y = 0,$$

$$V_y - \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} + m_y = 0.$$

Finally:

$$V_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - m_y. \tag{3}$$

**Rotational equilibrium with respect to $y$ axis.**

$V_x' \,\mathrm{d}y \,\mathrm{d}x - M_x' \,\mathrm{d}y + M_x \,\mathrm{d}y - M_{yx}' \,\mathrm{d}x + M_{yx} \,\mathrm{d}x + p(x,y) \,\mathrm{d}x \,\mathrm{d}y \,\frac{\mathrm{d}x}{2} + m_x \,\mathrm{d}y \,\mathrm{d}x = 0.$

The term $p(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\frac{\mathrm{d}x}{2}$ is dropped out since it represents an infinitesimal of higher order.

$$\left(V_x + \frac{\partial V_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}x - \left(M_x + \frac{\partial M_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y + M_x\,\mathrm{d}y \\ - \left(M_{yx} + \frac{\partial M_{yx}}{\partial y}\,\mathrm{d}y\right)\mathrm{d}x + M_{yx}\,\mathrm{d}x + m_x\,\mathrm{d}y\,\mathrm{d}x = 0,$$

$$V_x - \frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} + m_x = 0.$$

Finally:

$$V_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - m_x. \tag{4}$$

**Translational equilibrium.**

$$\frac{\partial V_x}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + \frac{\partial V_y}{\partial y}\,\mathrm{d}y\,\mathrm{d}x + p(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + p(x,y) = 0. \tag{5}$$

**Plate equilibrium equation.** Substituting equations (3) and (4) into equation (5) one can work out the equilibrium equation of the rectangular plate element:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x,y) - \frac{\partial m_y}{\partial y} - \frac{\partial m_x}{\partial x} = 0 \tag{6}$$

**Generalized constitutive relationship.** In order to develop the generalized constitutive relationship, the definition of elastic specific energy per unit area will be exploited. Recalling the local constitutive relationship $\underline{\sigma} = \underline{\underline{D}}\,\underline{\varepsilon}$, where $\underline{\underline{D}}$ is the stiffness matrix:

$$\underline{\underline{D}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2} \end{bmatrix},$$

one can write down the elastic specific energy per unit area:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}A} = \frac{1}{2}\int_{-h/2}^{+h/2} \underline{\varepsilon}^T \underline{\sigma}\,\mathrm{d}z = \frac{1}{2}\underline{q}^T \int_{-h/2}^{+h/2} \underline{\underline{b}}^T \underline{\underline{D}}\,\underline{\underline{b}}\,\mathrm{d}z\,\underline{q} = \frac{1}{2}\underline{q}^T \underline{\underline{D}}^* \underline{q}.$$

From the last expression it is clear that the generalized stiffness matrix $\underline{\underline{D}}^*$ is equal to:

$$\begin{aligned} \underline{\underline{D}}^* &= \int_{-h/2}^{+h/2} \underline{\underline{b}}^T \underline{\underline{D}}\,\underline{\underline{b}}\,\mathrm{d}z \\ &= \int_{-h/2}^{+h/2} \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{d}z. \end{aligned}$$

Remembering that the moment of inertia of a unit length element is given by:

$$\int_{-h/2}^{+h/2} 1 \cdot z^2\,\mathrm{d}z = I,$$

one can finally work out $\underset{=}{D}^*$ as follows:

$$\underset{=}{D}^* = \frac{EI}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \left(\dfrac{1-\nu}{2I}\right)h & 0 \\ 0 & 0 & 0 & 0 & \left(\dfrac{1-\nu}{2I}\right)h \end{bmatrix}.$$

The generalized stiffness matrix $\underset{=}{D}^*$ just obtained relates the generalized stresses $\underset{-}{Q}$ to the generalized strains $\underset{-}{q}$:

$$\underset{-}{Q} = \underset{=}{D}^* \underset{-}{q}, \tag{7}$$

where $\underset{-}{q}$ and $\underset{-}{Q}$ are the vectors:

$$\underset{-}{q} = \begin{bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ t_x \\ t_y \end{bmatrix},$$

$$\underset{-}{Q} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \\ V_x \\ V_y \end{bmatrix}.$$

Equation (7) can be rewritten in expanded form, giving rise to the following relationships:

$$M_x = D(\chi_x + \nu\chi_y),$$
$$M_y = D(\chi_y + \nu\chi_x),$$
$$M_{xy} = D\frac{1-\nu}{2}\chi_{xy} = \frac{EI}{1-\nu^2}\frac{1-\nu}{2}\chi_{xy} = \frac{EI}{2(1+\nu)}\chi_{xy} = GI\chi_{xy},$$
$$V_x = G\,h\,t_x,$$
$$V_y = G\,h\,t_y,$$

where:

- $D = \dfrac{EI}{1-\nu^2}$ is the flexural rigidity factor, which includes all the elastic constants related to material;
- $G = \dfrac{E}{2(1+\nu)}$ is the shear modulus;
- $I = \dfrac{1 \cdot h^3}{12}$ is the moment of inertia of a unit length element.

# 3 Thin plates theory

If a plate is thin enough with respect to its height, it is possible to neglect the shear deformations. Usually it is considered that a plate falls into this hypothesis field if $h < \min(a, b)/5$, where $h$ is the thickness, $a$ and $b$ are the other two dimensions. Furthermore, also the condition that the maximum displacement of the plate must be smaller than 1/5 of the thickness should be satisfied (Belluzzi, 1966). If the previous conditions are met, then the generic straight segment initially perpendicular to the middle plane remains perpendicular to it even after the deformation. This removes the possibility of having angular (i.e., shear) deformations. This hypothesis was first studied by Kirchhoff and it is usually named after him (Timoshenko and Woinowsky-Krieger, 1959). The Kirchhoff's hypothesis can be represented by the following mathematical condition:

$$\gamma_{xz} = \gamma_{yz} = 0,$$

which implies (see equation (1)):

$$\varphi_x = \frac{\partial w}{\partial x}, \qquad \varphi_y = \frac{\partial w}{\partial y}.$$

Now it is clear from the previous expressions that, under the Kirchhoff's hypothesis, the rotation of the generic straight segment is exactly equal to the one of the middle plane, meaning that there are no angular deformations. Therefore the plate model can be reformulated in this simplified case, obtaining the expressions reported below.

Local displacement vector:

$$\underline{s} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -z\varphi_x \\ -z\varphi_y \\ w(x,y) \end{bmatrix} = \begin{bmatrix} -zw_{,x} \\ -zw_{,y} \\ w(x,y) \end{bmatrix}.$$

Generalized displacement vector:

$$\underline{U} = \begin{bmatrix} w \\ w_{,x} \\ w_{,y} \end{bmatrix}.$$

Generalized strain vector:

$$\underline{q} = \begin{bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{bmatrix}.$$

Local strain vector:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} z\chi_x \\ z\chi_y \\ z\chi_{xy} \end{bmatrix} = \begin{bmatrix} -zw_{,xx} \\ -zw_{,yy} \\ -2zw_{,xy} \end{bmatrix}.$$

The generalized constitutive relationships give rise to the following expressions:

$$M_x = D(\chi_x + \nu\chi_y) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \tag{8}$$

$$M_y = D(\chi_y + \nu\chi_x) = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{9}$$

$$M_{xy} = GI\chi_{xy} = -D(1-\nu)\left(\frac{\partial^2 w}{\partial x\partial y}\right). \tag{10}$$

Recalling the plate equilibrium equation (6) and neglecting the terms related to distributed micro-couples:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x,y) = 0. \tag{11}$$

Substituting the equations (8), (9) and (10) into equation (11) one can get:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{p(x,y)}{D},$$

namely:

$$\boxed{\nabla^4 w(x,y) = -\frac{p(x,y)}{D},} \tag{12}$$

which is the Germain–Lagrange equation for thin plates (i.e., plates under the Kirchhoff hypothesis). It should be noticed that this equation includes in itself the equilibrium condition, the compatibility equation and the constitutive relationship. It appears as a generalization to the two-dimensional case of the unidimensional Euler–Bernoulli equation for beams (Timoshenko and Woinowsky-Krieger, 1959).
In equation (12) appears the symbol $\nabla^4$ which represents the Laplacian operator of fourth order. The Laplacian of a function allows to compare the function at a point with the function at neighbouring points (Farlow, 1993). The Laplacian of fourth order can be viewed as a generalization of the unidimensional fourth derivative to higher dimension.

## 3.1 Circular plates

Since in the present work only circular plates will be analysed, it is convenient to express the governing differential equation in polar coordinates, which can be easily achieved by performing a coordinate transformation. Figure 6 illustrate the equilibrium of a circular plate element.

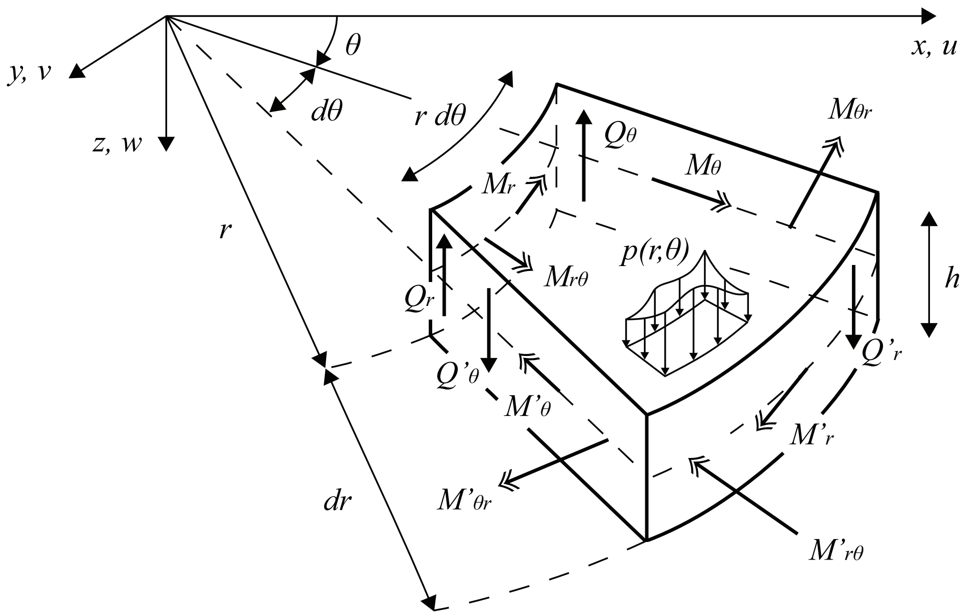


Figure 6: Equilibrium of a circular plate element.

The geometrical relationships between Cartesian and polar coordinates are:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \theta = \arctan\left(\frac{y}{x}\right),$$

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2}}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta,$$

$$\frac{\partial r}{\partial y} = \frac{\partial\sqrt{x^2+y^2}}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta,$$

$$\frac{\partial \theta}{\partial x} = \frac{\arctan(y/x)}{\partial x} = -\frac{y/x^2}{1+(y/x)^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r},$$

$$\frac{\partial \theta}{\partial y} = \frac{\arctan(y/x)}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

Applying the chain rule:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial w}{\partial \theta}\sin\theta.$$

Now it should be noted that for an axis-symmetric problem, like all the ones that will be treated in the present work, holds:

$$\frac{\partial}{\partial\theta} = 0,$$

i.e., all the terms involving partial derivatives with respect to $\theta$ can be dropped out. Therefore the previous expression can be simplified:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial w}{\partial r}\cos\theta.$$

To evaluate the term $\partial^2 w/\partial x^2$ the previous operation must be repeated twice, obtaining:

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial r^2}\cos^2\theta + \frac{\partial w}{\partial r}\frac{\sin^2\theta}{r}.$$

Analogously:

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2}\sin^2\theta + \frac{\partial w}{\partial r}\frac{\cos^2\theta}{r},$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial^2 w}{\partial r^2}\frac{\sin 2\theta}{2} - \frac{\partial w}{\partial r}\frac{\sin 2\theta}{2r}.$$

Adding term by term:

$$\nabla_r^2 w \equiv \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}.$$

Repeating the operation twice, one can get the governing differential equation for axis-symmetric plates in polar coordinates:

$$\nabla_r^4 w(r,\theta) \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = \frac{p(r,\theta)}{D}.$$

Since the plate geometry is symmetric and also the load distribution will be assumed to be axis-symmetric throughout this work, the previous equation can be simply rewritten as:

$$\nabla_r^4 w(r) = \frac{p(r)}{D}. \tag{13}$$

From the expressions outlined above the curvatures in polar coordinates can be worked out (assuming that $x$ axis is taken in the direction of the radius $r$, at $\theta = 0$, in order to simplify the derivations):

$$\chi_x = \chi_r = -\frac{\partial^2 w}{\partial x^2} = -\frac{\partial^2 w}{\partial r^2},$$

$$\chi_y = \chi_\theta = -\frac{\partial^2 w}{\partial y^2} = -\frac{1}{r}\frac{\partial w}{\partial r},$$

$$\chi_{xy} = \chi_{r\theta} = -\frac{\partial^2 w}{\partial x \partial y} = 0.$$

Now the relationships between moment and curvatures:

$$M_r = M_x = D(\chi_x + \nu\chi_y) = -D\left(\frac{\partial^2 w}{\partial r^2} + \nu\frac{1}{r}\frac{\partial w}{\partial r}\right),$$

$$M_\theta = M_y = D(\chi_y + \nu\chi_x) = -D\left(\frac{1}{r}\frac{\partial w}{\partial r} + \nu\frac{\partial^2 w}{\partial r^2}\right),$$

$$M_{r\theta} = M_{xy} = D(1-\nu)\chi_{xy} = 0.$$

Elastic strain energy computation:

$$\begin{aligned}
U &= \frac{1}{2}\iint\limits_S (M_x\chi_x + M_y\chi_y + 2M_{xy}\chi_{xy})\,\mathrm{d}S,\\
&= \frac{1}{2}\iint\limits_S (D(\chi_x + \nu\chi_y)\chi_x + D(\chi_y + \nu\chi_x)\chi_y + 2D(1-\nu)\chi_{xy}^2)\,\mathrm{d}S,\\
&= \frac{1}{2}D\iint\limits_S (\chi_x^2 + \chi_y^2 + 2\nu\chi_x\chi_y + 2(1-\nu)\chi_{xy}^2)\,\mathrm{d}S,\\
&= \frac{1}{2}D\iint\limits_S (\chi_x^2 + \chi_y^2 + 2\chi_x\chi_y - 2\chi_x\chi_y + 2\nu\chi_x\chi_y + 2(1-\nu)\chi_{xy}^2)\,\mathrm{d}S,\\
&= \frac{1}{2}D\iint\limits_S ((\chi_x + \chi_y)^2 - 2\chi_x\chi_y(1-\nu) + 2(1-\nu)\chi_{xy}^2)\,\mathrm{d}S,\\
&= \frac{1}{2}D\int_0^{2\pi}\int_0^R \left[(\chi_x + \chi_y)^2 - 2(1-\nu)(\chi_x\chi_y - \chi_{xy}^2)\right]\, r\,\mathrm{d}\theta\,\mathrm{d}r.
\end{aligned}$$

Finally, substituting the expressions of the curvatures into the last equation:

$$U = \frac{1}{2}D\int_0^{2\pi}\int_0^R \left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial r^2}\frac{1}{r}\frac{\partial w}{\partial r}\right)\right] r\,\mathrm{d}\theta\,\mathrm{d}r. \quad (14)$$

This result can also be found in Clough and Penzien (1993). Since, as was previously mentioned, the plate deflection shape does not depend on $\theta$, the plate equation (13) can be rewritten in terms of total derivatives:

$$\nabla_r^4 w(r) \equiv \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}^2 w}{\partial r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = \frac{p(r)}{D}. \quad (15)$$

Introducing the identity:

$$\nabla_r^4 w(r) \equiv \frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)$$

Equation (15) now becomes:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]\right\} = \frac{p(r)}{D}. \tag{16}$$

The solution of this equation is given by a sum of the solution of the associated homogeneous differential equation $w_h$ and the particular solution $w_p$:

$$w = w_h + w_p.$$

The solution of the associated homogeneous form of (16) is worked out:

$$w_h = C_1 \ln r + C_2 r^2 \ln r + C_3 r^2 + C_4,$$

where $C_1$, $C_2$, $C_3$ and $C_4$ are constants that can be evaluated from the boundary conditions. The particular solution $w_p$ is obtained by successive integration of equation (16):

$$w_p = \int \frac{1}{r} \int r \int \frac{1}{r} \int \frac{r\, p(r)}{D}\, \mathrm{d}r\, \mathrm{d}r\, \mathrm{d}r\, \mathrm{d}r.$$

If the slab is subjected to a uniform distributed load with intensity constant in the radial direction equal to $p(r) = p_0$, the particular solution is:

$$w_p = \frac{p_0 r^4}{64D}.$$

Therefore the general solution of equation (16) is:

$$w(r) = C_1 \ln r + C_2 r^2 \ln r + C_3 r^2 + C_4 + \frac{p_0 r^4}{64D},$$

$$M_r = -D\left[C_1\frac{1-\nu}{r^2} + 2C_2(1+\nu)\ln r + C_2(3+\nu) + 2C_3(1+\nu) + \frac{p_0 r^2}{16D}(3+\nu)\right].$$

Particular cases of boundary conditions must be considered in order to determine the four constants.

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